THEORY OF ARTERIAL CIRCULATION

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UNIVERSITY OF DAYTON RESEARCH INSTITUTE

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13 ABSTRACT

This report is a unification and elucidation of theories of arterial circulation and is based, particularly, on Womersley's extensive work: "An Elastic Tube Theory of Pulse Transmission and Oscillatory Flow in Mammalian Arteries." In an analytical description of the arterial circulation, the concept of a thin-walled elastic tube filled with a viscous fluid as a rough working model of an artery is used. The representative equations are linearized and periodic solutions are obtained for various flow parameters that can be tested experimentally. The main topic is the mode of pulse-wave transmission, and the relationships between pulse pressure, rate of flow and radial expansion in the artery. Moreover, the significance of a salient non-dimensional parameter which is a function of the frequency, the kinematic viscosity of the fluid and the internal radius of the tube is stressed throughout in characterizing the motion of the fluid. Some comparisons with experimental results are made, and new experiments are proposed, as tests of the adequacy of the theory.

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SUMMARY

The problems treated in this report are those forming the main theme of Womersley's theory of arterial circulation, which pertains to the blood flow in the arteries based upon the differential equations of liquid flow in a thin-walled elastic tube. In particular, the problems dealt with are those relating to: (1) wave propagation in the arterial system; (2) pulsatile pressure and flow changes associated with the wave propagation; and (3) relationship between pulsatile pressure on the one hand and the geometry and the physical properties of the arterial system on the other.

Some of the results which follow from the quantitative relationships of this theory are:

(1) Changes in the viscoelastic properties of the arterial wall are important with regard to wave propagation.

(2) The flow generated by a given oscillatory pressure gradient does not vary greatly over a wide range of changes in additional tissue mass and elastic constraints of the tube.

(3) The phase difference between periodic variations in pressure and tube diameter is also insensitive to a wide range of variations in tissue mass and elastic constraint.

Womersley's work indicates that the thin-walled elastic tube can be used as a rough working model of the artery. Moreover, according to this model a number of relationships between observable quantities such as flow and pressure gradient, and between pressure and tube diameter can be deduced and verified experimentally.

FOREWORD

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This technical report has been reviewed and is approved.

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NOTATION

$A, A_1, \ldots, A_m, \ldots$	complex constants defining amplitude and phase of pressure or pressure gradient, according to context
$B = E/(1 - \sigma^2)$	
C_1, D_1, E_1	arbitrary constants
c	complex velocity of wave propagation
c_0	velocity of wave propagation for a fluid of zero viscosity
c_1	velocity of wave propagation
c_g	group velocity
E	Young's modulus of tube material
E_c	complex constant replacing Young's modulus when there is internal damping in the tube wall
$E(\ell,m)$	standard correction for finite expansion of the tube, expressing the effect on the average velocity of the $(\ell+m)^{th}$ harmonic of interaction between the ℓ^{th} and m^{th} harmonics
$E(m,0)$	as above, for the interaction between the m^{th} harmonic and the steady flow
$E(m,-m)$	as above, expressing the effect of the m^{th} harmonic on the steady flow
f	frequency in cycles per second
$F_{10}(\alpha) = \dfrac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})}$	
h	tube wall thickness
$h_0 = \dfrac{M_0(\alpha y)}{M_0(\alpha)}$	
$h_{10} = \dfrac{2M_1(\alpha)}{\alpha M_0(\alpha)}$	

$k = h/R$	ratio of wall thickness to radius of tube
M	modulus of applied pressure gradient
$M_0(x) = \lvert J_0(xi^{3/2}) \rvert$	
$M_1(x) = \lvert J_1(xi^{3/2}) \rvert$	
$M_0'(\alpha y) = \left\lvert 1 - \frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})} \right\rvert = \left\lvert 1 - h_0 e^{-i\delta_0} \right\rvert$	
$M_{10}'(\alpha) = \left\lvert 1 - \frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})} \right\rvert = \left\lvert 1 - h_{10} e^{-i\delta_{10}} \right\rvert$	
$M_{10}'' = \lvert 1 + \eta F_{10}(\alpha) \rvert$	
$n = 2\pi f$	circular frequency
$q = \int_0^y w(2y)\,dy$	average velocity over a cross section of radius y, $y < 1$
Q	volume rate of flow
$Q_{max} = \lvert Q \rvert$	maximum value of Q
Q_{steady}	Poiseuille flow corresponding to maximum value of pressure gradient, if maintained constant
r	radial coordinate
R	radius of tube
t	time
u	radial component of fluid velocity
w	longitudinal component of fluid velocity
w_0	steady component of longitudinal velocity
w_0	(section V) value of w at $y=y_0$
$\bar{w} = \int_0^1 w(2y)\,dy$	average velocity across the tube
$\bar{w}_0$	average velocity across the tube of the steady stream

$W(\ell,m)$, $W(m,0)$, $W(m,-m)$ standard corrections for the effect of the quadratic terms in the Navier-Stokes equations

$x = \frac{hB}{R\rho_0 c^2}$

x (Section X) $x = (h\sigma i^{3/2})y^2$

X real part of c_0/c

Y imaginary part of c_0/c

$y = r/R$, nondimensional radial coordinate

z distance along axis of tube

Subscript m quantity corresponding to the m^{th} harmonic

Superscript * complex conjugate of quantity

$\alpha = (\frac{R^2 n}{\nu})^{1/2}$ nondimensional frequency parameter

β (Section V), $\beta = \alpha(\frac{\mu}{\mu_0})^{1/2}$ nondimensional parameter of the motion in the plasma layer

β (Section X), $\beta = \alpha(1 - b^2)^{1/2} = \alpha\left|1 - (\frac{2\bar{w}_0}{c})^2\right|^{1/2}$

β_0 (Section X), $\beta_0 = \alpha\left|1 - (\frac{2\bar{w}_0}{c_0})^2\right|^{1/2}$

$\gamma = (\frac{1}{b^2} - 1)b\alpha i^{1/2}$

$\delta_0 = \Theta_0(\alpha) - \Theta_0(\alpha y)$

$\delta_{10} = 135° - \gamma_1 + \Theta_0$

$\varepsilon_0'(\alpha y) = \text{phase of } \left|1 - \frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})}\right| = \text{phase of } [1 - h_0 e^{-i\delta_0}]$

$\varepsilon_{10}'(\alpha) = \text{phase of } \left|1 - \frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})}\right| = \text{phase of } [1 - h_{10} e^{-i\delta_{10}}]$

$\varepsilon_{10}'' = \text{phase of } [1 + \eta F_{10}(\alpha)]$

ζ	longitudinal displacement of tube wall
η	complex constant appearing in flow formula
Θ	circumferential coordinate
$\Theta_0(x)$	phase of $J_0(xi^{3/2})$
$\Theta_1(x)$	phase of $J_1(xi^{3/2})$
λ	refer to equation 7-10
μ	viscosity of the fluid
$\nu = \frac{\mu}{\rho_0}$	kinematic viscosity of the fluid
ξ	radial displacement of tube wall
ρ	density of tube material
ρ_0	density of the fluid
σ	Poisson's ratio of tube material
σ_c	complex constant replacing Poisson's ratio when there is internal damping in the tube wall
ϕ	negative phase of applied pressure gradient
ψ	phase of fluid pressure

SECTION I

INTRODUCTION

The mammalian circulatory system is essentially a fluid transport system. An important part of this system is the arterial tree, which may be considered as a branching conduit system having the function of delivering blood to the tissues with a minimum loss of energy.

In an analytic description of the arterial circulation, the investigation consists in determining the characteristics of a system composed of a non-Newtonian fluid flowing within a branching system of tapered, distensible tubes and subjected to phasic changes in pressure. The distribution of pulsatile pressure and flow at various locations in the system is modified by a number of factors, and is therefore difficult to describe and predict. Some of these factors are:

(1) The transient phenomena due to the mechanical action of the heart.

(2) The branching, tapering and tethering of the blood vessels.

(3) The impedance provided by the arterioles.

Application of mathematical and physical principles by several investigators over the past two hundred years have contributed significantly to a better understanding of the hemodynamic aspects of the cardiovascular system, the development of special instrumentation, and the evaluation of experimental records. A highly useful mathematical approach to this formidable problem was developed primarily by J. R. Womersley and his co-workers D. A. McDonald and M. G. Taylor.

The main value of Womersley's work lies, it is believed, in its endeavor to outline in a simple manner the analysis of the circulation as a system in steady-state oscillation, based on standard principles of fluid dynamics. The equations of state for both the blood and the blood-vessel system are set up, the equations are linearized, and periodic solutions in the form of Fourier series which satisfy prescribed boundary conditions are developed. In particular, his work indicates that the thin-walled elastic tube can be used as a rough working model of the artery. Moreover, from this model a number of relationships between observable quantities can be deduced and tested experimentally. Womersley's theory does not take into consideration significant taper in the tube system or nonuniformity of the physical properties of the blood vessels.

The problems that form the main theme of Womersley's work, described in this report, pertain to the flow in the arteries and, in particular, are those concerned with (1) the velocity of wave propagation; (2) the pulsatile flow and pressure changes associated with the wave propagation; and (3) the pulsatile pressure-diameter relationships. Moreover, the significance of

a salient nondimensional parameter, denoted by α, which is a function of the frequency, the kinematic viscosity of the fluid and the internal radius of the tube, is stressed in characterizing the motion of the fluid. The variation of this parameter at corresponding flow points in mammals is very small and could therefore be considered as a Reynolds number for pulsatile flow.

Womersley's work forms an important link in the continuing chain of understanding. We have chosen to present his version not because it is the most sophisticated work in this area but because within its limitations it is a well-developed treatment of several aspects of the arterial problem, and suggests a rational basis for many of the peculiar characteristics observed in the mammalian cardiovascular system. Moreover, it has indicated directions for further improvement in the mathematical analysis of the cardiovascular system and has encouraged experimental investigations along these lines.

In this report section II begins with a highly idealized model of the arterial system, the linearized flow of a viscous, incompressible fluid in a straight, rigid, circular tube, in order to develop the basic concepts of the problem. This model is then successively refined in order to study the effects of the elasticity of the tube, the oscillatory changes in tube diameter, the boundary layer near the walls of the arteries, the junctions and discontinuities in the arterial tube system and finally, to assess approximately the effect of the nonlinear terms in the flow equations.

In its original form, Womersley's work is understandable only to specialists in this particular area of research. In this present, expanded form, we believe that his work would be accessible as well as of interest to a much larger audience who are interested in hemodynamics from the experimental as well as the analytical point of view.

SECTION II

OSCILLATORY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A STRAIGHT, RIGID, CIRCULAR TUBE

INTRODUCTION

In this section we shall consider a very simplified model of the arterial system, which consists essentially of the laminar flow of a viscous, incompressible, Newtonian fluid in an infinitely long, uniform, rigid cylindrical tube. Such a system is characterized in terms of the Navier-Stokes equations. From these general equations, we shall, under the prescribed conditions, derive the equations describing the particular flow process of interest and obtain a solution. Moreover, we shall consider the limiting and modified forms of the solution equation and draw some conclusions.

Next, an expression for the volume rate of flow will be determined, and electrical analogues of flow quantities considered. In addition, the Fourier series representation for calculating the volume rate of flow will be obtained in terms of the pressure gradient.

Finally, the relationship between pressure gradient and the time rate of change of pressure will be discussed.

DERIVATION AND SOLUTION OF THE EQUATION DEFINING THE OSCILLATORY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A STRAIGHT, RIGID, CIRCULAR TUBE

The equations governing the laminar flow of a viscous incompressible fluid, expressed in cylindrical coordinates (see figure 1), are (Schlichting, 1960):

The equation of continuity of mass

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \qquad (2-1)$$

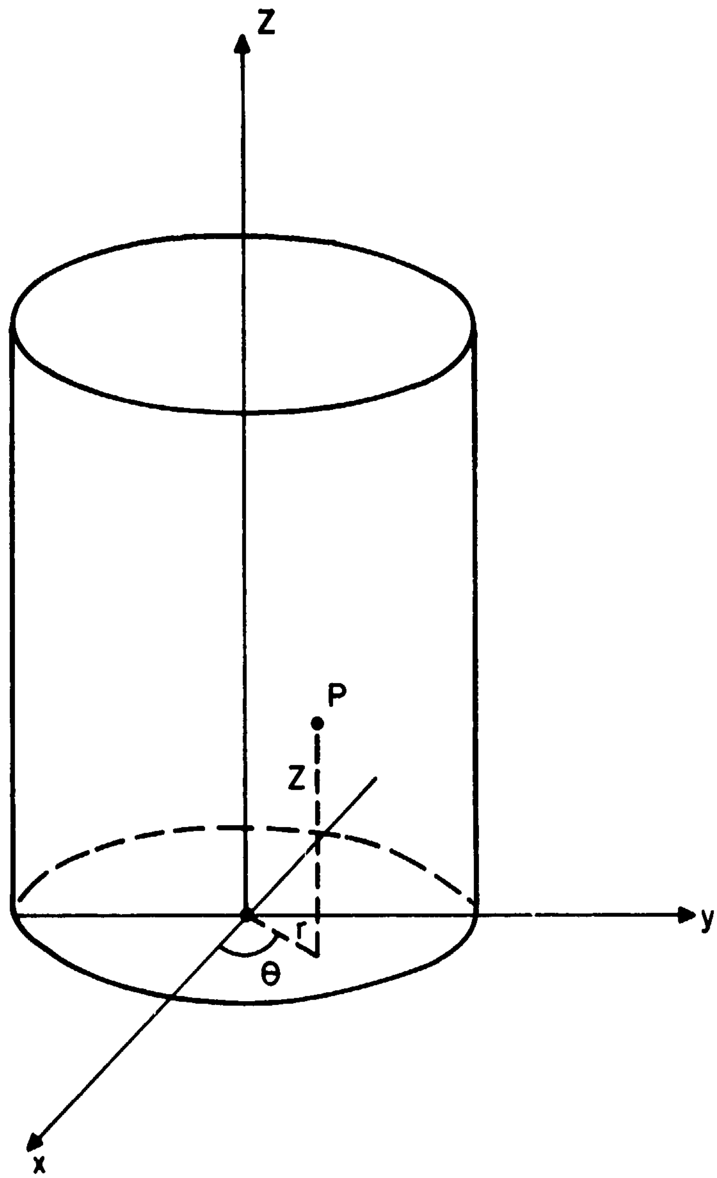


Figure 1. Cylindrical Coordinates of a Point Within the Flow Along the Z Axis

The three dynamical equations of motion

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right)$$

$$= F_r - \frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (2\text{-}2)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + w\frac{\partial v}{\partial z}\right)$$

$$= F_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial z^2}\right) \qquad (2\text{-}3)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right)$$

$$= F_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (2\text{-}4)$$

The assumptions made regarding the particular flow process under consideration through a straight, rigid, circular tube are as follows. (See figure 2.)

a. The radial and tangential motions of the fluid are neglected, $u = 0$, $v = 0$.

b. The fluid velocity along the axis of the tube (the z axis) is independent of the distance z, $\frac{\partial w}{\partial z} = 0$, i.e., the value of w remains unchanged along the tube axis.

c. w is a function of the radial coordinate, r, and time, t, $w = w(r,t)$.

d. The fluid is subjected to a longitudinal periodic pressure gradient $\frac{\partial p}{\partial z}$ having the form

$$-\frac{\partial p}{\partial z} = A\,e^{int} = A\,(\cos nt + i \sin nt) \qquad (2\text{-}5)$$

where A is a complex constant denoting the magnitude of the pressure gradient and ω = nt is the phase. The pressure gradient along the radial (r) and circumferential (θ) directions are zero.

e. The body force $F = (F_r, F_\theta, F_z)$ is neglected, i.e., $F_r = F_\theta = F_z = 0$.

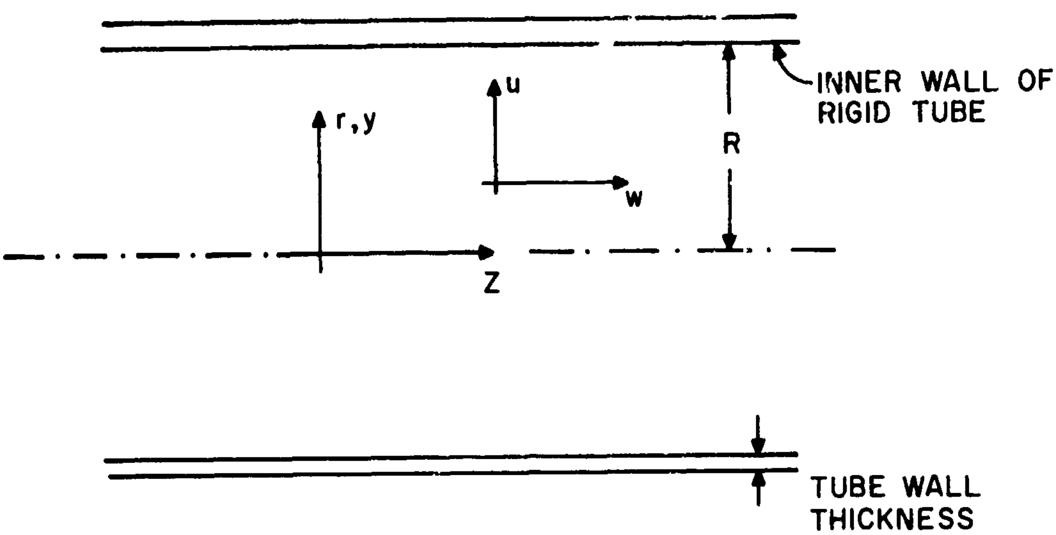


Figure 2. Coordinate System

If we impose the restrictions as specified in the assumptions (a), (b), (c) and (e) above, we find that all the terms in the general flow equations (2-1, 2-2, and 2-3) vanish. We are left with the following terms of equation 2-4

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

or

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{2-6}$$

Equation 2-6 defines the flow process under investigation without the imposition of the periodic pressure gradient as specified in equation 2-5.

Since the longitudinal pressure gradient has the form described by equation 2-5, it follows that the longitudinal fluid velocity w, subject to this pressure gradient, may be considered to have the form

$$w = w_1 \, e^{int} \tag{2-7}$$

where w_1 denotes the magnitude of the fluid velocity. Since, according to assumption c, w is a function of r and t, we write equation 2-7 more precisely as

$$w = w(r,t) = w_1(r) \, e^{int} \tag{2-8}$$

Now we combine equations 2-5 and 2-8 with equation 2-6. We note that

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[w_1(r) \, e^{int} \right] = w_1(r) \, in \, e^{int}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{1}{\rho} A e^{int}$$

$$\frac{\nu}{r}\frac{\partial w}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left[w_1(r) e^{int} \right] = \frac{\nu}{r}\frac{dw_1}{dr} e^{int}$$

$$\nu \frac{\partial^2 w}{\partial r^2} = \nu \frac{d^2 w_1}{dr^2} e^{int}$$

Since w_1 is a function of r alone, we replace the partial derivative notation by the total derivative notation. Thus, equation 2-6 has the form

$$w_1 \, in \, e^{int} = \frac{A}{\rho} e^{int} + \nu \frac{d^2 w_1}{dr^2} e^{int} + \frac{\nu}{r}\frac{dw_1}{dr} e^{int}$$

or $$w_1 in = \frac{A}{\rho} + \nu \frac{d^2 w_1}{dr^2} + \frac{\nu}{r}\frac{dw_1}{dr}$$

or $$\frac{d^2 w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr} - \frac{in}{\nu} w_1 = -\frac{A}{\mu} \tag{2-9}$$

We now write equation 2-9 in terms of a new independent (and nondimensional) variable $y = r/R$. Accordingly, the first two terms in equation 2-9 may be written as

$$\frac{1}{r}\frac{dw_1}{dr} = \frac{1}{Ry}\frac{dw_1}{d(Ry)} = \frac{1}{R^2 y}\frac{dw_1}{dy}$$

$$\frac{d^2 w_1}{dr^2} = \frac{d}{dr}\left[\frac{dw_1}{dr}\right] = \frac{d}{d(Ry)}\left[\frac{dw_1}{d(Ry)}\right] = \frac{1}{R^2}\frac{d^2 w_1}{dy^2}$$

Thus, in terms of the independent variable y, equation 2-9 has the form

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} - \frac{inR^2}{\nu} w_1 = -\frac{AR^2}{\mu} \tag{2-10}$$

We observe that the physical parameters R, n and ν which characterize the motion of the fluid appear together as a product in the form R^2n/ν in equation 2-10. For convenience, we denote this product by α^2. Since the values of R, n and ν are always positive, we use α^2 (Instead of α) to emphasize that the product R^2n/ν is always positive.

Note that $\alpha^2 = R^2n/\nu$ is a dimensionless parameter,

$$[\alpha^2] = \left[\frac{R^2n}{\nu}\right] = \frac{L^2\left(\frac{1}{T}\right)}{L^2/T} = 1$$

Here, L and T denote dimensions of length and time. Since the value of α depends upon the frequency n, we may say that α is a dimensionless frequency parameter. We may also write

$$\alpha^2 = \frac{R^2n}{\nu} = \frac{\rho R^2n^2}{\rho\nu n} = \frac{\text{magnitude of typical oscillatory pressure force}}{\text{magnitude of typical oscillatory viscous force}} .$$

Thus α may be considered as an oscillatory Reynolds number. If $\alpha >> 1$, then the flow may be considered as inviscid.

Setting $\alpha^2 = R^2n/\nu$ in equation 2-10, we have

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} - i\alpha^2 w_1 = -\frac{AR^2}{\mu}$$

or
$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = -\frac{AR^2}{\mu} \qquad (2\text{-}11)$$

Equation 2-11 defines the particular flow process under investigation. The problem now is to determine the solution of equation 2-11, having the form $w_1 = w_1(y)$, satisfying the specific boundary conditions to be imposed and containing the flow parameters A, μ, ν, n and R.

Equation 2-11 is a nonhomogeneous Bessel differential equation. The corresponding homogeneous differential equation is

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = 0 \qquad (2\text{-}12)$$

The solution of equation 2-12 may be written in the form (Watson, 1944):

$$w_1(y) = K_1 J_0(i^{3/2}\alpha y) \tag{2-13}$$

which is known as the complementary function. K_1 is an arbitrary constant to be evaluated. For the nonhomogeneous equation 2-11, we let

$$w_1(y) = K_2 = \text{constant} \tag{2-14}$$

Substituting equation 2-14 in equation 2-11, we obtain

$$i^3\alpha^2 K_2 = -\frac{AR^2}{\mu}$$

i.e.

$$K_2 = \frac{AR^2}{i\alpha^2\mu} \tag{2-15}$$

Equation 2-15 is the particular solution of equation 2-11. Thus, the complete solution of equation 2-11 is

$$w_1(y) = K_1 J_0(i^{3/2}\alpha y) + \frac{AR^2}{i\alpha^2\mu} \tag{2-16}$$

To evaluate the constant K_1, we impose the condition of "no slip" at the tube wall $r = R$:

$$w = 0 \quad \text{at} \quad y = \frac{r}{R} = 1.$$

Imposing this condition on equation 2-16, we obtain

$$K_1 J_0\left(i^{3/2}\alpha \cdot 1\right) + \frac{AR^2}{i\alpha^2\mu} = 0$$

or

$$K_1 = -\frac{AR^2}{i\alpha^2\mu\, J_0\left(i^{3/2}\alpha\right)}$$

Rewriting equation 2-16 in terms of this value of K_1 we have

$$w_1(y) = -\frac{AR^2 J_0(i^{3/2}\alpha y)}{i\alpha^2\mu\, J_0(i^{3/2}\alpha)} + \frac{AR^2}{i\alpha^2\mu}$$

$$= \frac{AR^2}{i\alpha^2\mu}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right] \tag{2-17}$$

Combining equations 2-7 and 2-17, we obtain the fluid velocity along the axis of the tube

$$w = w(y,t) = w_1(y)e^{int} = \frac{AR^2}{i\alpha^2\mu}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right]e^{int} \tag{2-18}$$

The factor $AR^2/i\alpha^2\mu$, appearing in equation 2-18, may be simplified for computational purposes as follows. We first note that

$$\alpha^2 = \frac{R^2 n}{\nu} = \frac{R^2 n\rho}{\mu}$$

Thus $$\frac{AR^2}{i\alpha^2\mu} = \frac{A}{in\rho}$$

and $$w = w(y,t) = \frac{A}{in\rho}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right]e^{int} \tag{2-19}$$

A formula essentially the same as the real part of equation 2-19, when A is real, is given by Egami (1944) and Lambossy (1953). The latter has also developed a formula for the viscous drag. Lambossy and Thurston (1952), who also investigated the problem, were concerned with the effect of fluid resistance on the frequency-response of measuring instruments.

LIMITING FORMS OF THE SOLUTION OF THE EQUATION DEFINING THE OSCILLATORY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A STRAIGHT, RIGID, CIRCULAR TUBE

We note that the longitudinal fluid velocity, w, as described by equation 2-18, is a function of the nondimensional coordinate, y and time, t. We also observe that the value of w is dependent upon the value of the nondimensional frequency parameter $\alpha = \left(\frac{R^2 n}{\nu}\right)^{1/2}$. For fixed values of R and ν, the value of α varies directly as the value of $(n)^{1/2}$. So it is reasonable to look at the variation of the fluid velocity, w, for small and large values of α, i.e., for small and large values of $n = 2\pi f$, i.e., for small and large values of the frequency of oscillation, f, of the fluid.

If we include a phase lag between the oscillating pressure and the flow generated, then the pressure gradient imposed on the fluid has the form

$$-\frac{\partial p}{\partial z} = M e^{i(nt-\varphi)} = M\cos(nt-\varphi) + iM\sin(nt-\varphi) \tag{2-20}$$

instead of the form given by equation 2-5. Here, M is the magnitude of the pressure gradient and ϕ denotes the phase lag of the flow rate behind the pressure gradient. Accordingly, the fluid velocity, w, as described by equation 2-18, has the form

$$w = \frac{MR^2}{i\mu\alpha^2}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right]\left[\cos(nt-\varphi) + i\sin(nt-\varphi)\right] \tag{2-21}$$

We will now consider the limiting forms of equation 2-21, describing the fluid velocity for

Case I: Small values of the fluid parameter α, i.e., for small values of the frequency of oscillation of the flowing fluid.

Case II: Large values of the fluid parameter α, i.e., for large values of the frequency of oscillation of the flowing fluid.

Case I. First we consider the expansions

$$J_0(y) = 1 - \frac{y^2}{2^2} + \frac{y^4}{(2^2)(4^2)} - \frac{y^6}{(2^2)(4^2)(6^2)} + \cdots$$

$$J_0(iy) = 1 - \frac{i^2 y^2}{2^2} + \frac{(i^2)^2 y^4}{(2^2)(4^2)} - \frac{(i^2)^4 y^6}{(2^2)(4^2)(6^2)} + \cdots$$

$$= 1 + \frac{y^2}{2^2} + \frac{y^4}{(2^2)(4^2)} + \frac{y^6}{(2^2)(4^2)(6^2)} + \cdots$$

$$J_0(i^{3/2}\alpha y) = 1 - \frac{(i^{3/2}\alpha)^2 y^2}{2^2} + \frac{(i^{3/2}\alpha)^4 y^4}{(2^2)(4^2)} - \cdots$$

$$= 1 - \frac{i^3 \alpha^2 y^2}{2^2} + \frac{i^6 \alpha^4 y^4}{(2^2)(4^2)} - \cdots$$

$$= 1 - \frac{i^3 \alpha^2 y^2}{2^2} + \frac{i^6 (\alpha y)^4}{2^6} - \cdots$$

$$J_0\left(i^{3/2}\alpha\right) = 1 - \frac{i^3\alpha^2}{2^2} + \frac{i^6\alpha^4}{(2^2)(4^2)} - \cdots$$

$$= 1 - \frac{i^3\alpha^2}{2^2} + \frac{i^6\alpha^4}{(2)^6} - \cdots$$

Next, from the above expansicns we note that since $0 < y \leq 1$, for $(\alpha y)^4 << 2^6$, or for $\alpha << 2^{6/4}$, or for $\alpha << 3$, the values of $J_0(i^{3/2}\alpha y)$ an $J_0(i^{3/2}\alpha)$ may be written approximately as

$$J_0\left(i^{3/2}\alpha y\right) = 1 - \frac{i^3\alpha^2 y^2}{2^2} = 1 + \frac{i\alpha^2 y^2}{4}$$

$$J_0\left(i^{3/2}\alpha\right) = 1 - \frac{i^3\alpha^2}{2^2} = 1 + \frac{i\alpha^2}{4}$$

Accordingly, the term $\left| 1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \right|$ in equation 2-21 may be written as

$$1 - \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)} = 1 - \left(\frac{4 + i\alpha^2 y^2}{4 + i\alpha^2}\right) = \frac{i\alpha^2\left(1 - y^2\right)}{4 + i\alpha^2} = \frac{i\alpha^2\left(1 - y^2\right)}{4}$$

for small values of α. Thus, equation 2-21 has the approximate form

$$w \doteq \frac{MR^2}{i\alpha^2\mu}\left[\frac{i\alpha^2(1-y^2)}{4}\right]\left[\cos(nt-\varphi) + i\sin(nt-\varphi)\right]$$

$$= \frac{MR^2}{4\mu}\left[1-y^2\right]\left[\cos(nt-\varphi) + i\sin(nt-\varphi)\right]$$

$$= \frac{MR^2}{4\mu}\left[1-y^2\right]\cos(nt-\varphi) + \frac{iMR^2}{4\mu}\left[1-y^2\right]\sin(nt-\varphi) \qquad (2\text{-}22)$$

In equation 2-22, the part that has significance is the real part, i.e., the first term on the right-hand side. The imaginary part determines the phase of the fluid velocity. Thus for small values of α the fluid velocity, w, is a function of y and t and is given by

$$w = w(y,t) = \frac{MR^2}{4\mu}\left[1-y^2\right]\cos(nt-\varphi) \qquad (2\text{-}23)$$

In equation 2-23, if we consider that

a) there is no phase lag, i.e., $\phi = 0$;

b) the value of $n = 2\pi f$ is zero, i.e., the frequency, f, of the oscillating fluid is zero;

then $\cos(nt - \phi) - \cos 0^0 = 1$ and equation 2-23 reduces to the form

$$w = w(y) = \frac{MR^2}{4\mu}(1 - y^2) \qquad (2\text{-}24)$$

In equation 2-24, note that the dependence of the fluid velocity on time, t, has been eliminated, due to the restrictions $\phi = 0$ and $n = 0$. Moreover, since $y = r/R$,

$$1 - y^2 = 1 - \frac{r^2}{R^2} = \frac{R^2 - r^2}{R^2}$$

Substituting this value of $(1 - y^2)$ in equation 2-24 and simplifying, we obtain

$$w = w(r) = \frac{M}{4\mu} (R^2 - r^2) \qquad (2\text{-}25)$$

In equation 2-25, note that the fluid velocity, w, is a function of the radial coordinate, r, only.

Now, the equation describing the fluid velocity for stationary Poiseuille flow in a straight, rigid, circular tube is (Schlichting, 1960):

$$w = w(r) = \frac{p_1 - p_2}{4\mu L} (R^2 - r^2) \qquad (2\text{-}26)$$

where p_1 and p_2 are pressures at a distance, L, along the tube. Comparing equations 2-25 and 2-26, we find that the magnitude of the pressure gradient, M, corresponds to $\frac{p_1 - p_2}{L}$. Thus, equation 2-25 describes the fluid velocity for stationary Poiseuille flow in a straight, rigid, circular tube.

Case II. Again we consider equation 2-18. For large values of α, we shall use the asymptotic expansions for the expressions $J_0(i^{3/2}\alpha y)$ and $J_0(i^{3/2}\alpha)$. The asymptotic forms are

$$J_0\left(i^{3/2} y\right) \cong \frac{e^{\frac{y}{\sqrt{2}}}}{\sqrt{2\pi y}} \left[\cos\left(\frac{y}{\sqrt{2}} - \frac{\pi}{8}\right) + i \sin\left(\frac{y}{\sqrt{2}} - \frac{\pi}{8}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi y}} e^{\frac{y}{\sqrt{2}}(1+i) - \frac{i\pi}{8}}$$

$$J_0\left(i^{3/2}\alpha y\right) \cong \frac{1}{\sqrt{2\pi\alpha y}}\, e^{\frac{\alpha y}{\sqrt{2}}} \cdot e^{\frac{i\alpha y}{\sqrt{2}}} \cdot e^{-\frac{i\pi}{8}}$$

$$J_0\left(i^{3/2}\alpha\right) \cong \frac{1}{\sqrt{2\pi\alpha}}\, e^{\frac{\alpha}{\sqrt{2}}} \cdot e^{\frac{i\alpha}{\sqrt{2}}} \cdot e^{-\frac{i\pi}{8}} .$$

Accordingly, the factor $\left| 1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \right|$ in equation 2-18 may be written as

$$1 - \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)} \cong 1 - y^{-1/2} \cdot e^{\frac{1}{\sqrt{2}}\left(\alpha y - \alpha + i\alpha y - i\alpha\right)}$$

$$= 1 - y^{-1/2} \cdot e^{-\frac{\alpha}{\sqrt{2}}\left[(1-y)(1+i)\right]}$$

(2-27)

Thus, for large values of α, the fluid velocity, w, is obtained by combining equations 2-18 and 2-27

$$w = w(y,t) = \frac{AR^2}{i\mu\alpha^2}\left[1 - y^{-1/2} e^{-\frac{\alpha}{\sqrt{2}}\left[(1-y)(1+i)\right]}\right] e^{int}$$

(2-28)

If we consider the pressure gradient to be of the form given by equation 2-20 instead of the form given by equation 2-5, equation 2-28 may be written as

$$w = w(y,t) = \frac{MR^2}{i\mu\alpha^2}\left[1 - y^{-1/2} e^{-\frac{\alpha}{\sqrt{2}}\left[(1-y)(1+i)\right]}\right]\left[\cos(nt-\varphi) + i\sin(nt-\varphi)\right]$$

$$= \frac{MR^2}{i\mu\alpha^2}\left[1 - y^{-1/2} e^{-\frac{\alpha}{\sqrt{2}}(1-y)} e^{-\frac{i\alpha}{\sqrt{2}}(1-y)}\right]\left[\cos(nt-\varphi) + i\sin(nt-\varphi)\right]$$

(2-29)

For convenience of writing, we set $nt - \phi = D$ and $-\frac{\alpha}{\sqrt{2}}(1 - y) = E$. Equation 2-29 may then be written as

$$w = w(y,t) = \frac{MR^2}{i\mu\alpha^2}\left[1 - y^{-1/2} e^{E} e^{-iE}\right]\left[\cos D + i\sin D\right]$$

$$= \frac{MR^2}{i\mu\alpha^2}\left[\left(\cos D + i\sin D\right) - y^{-1/2} e^{E}\left\{\left(\cos E + i\sin E\right)\left(\cos D + i\sin D\right)\right\}\right]$$

$$= \frac{MR^2}{i\mu\alpha^2}\left[\left(\cos D + i\sin D\right) - y^{-1/2} e^{E}\left\{\cos D\cos E + i\sin D\cos E + i\cos D\sin E + i^2\sin D\sin E\right\}\right]$$

(2-30)

Considering only the real part of the fluid velocity, w, described by equation 2-30, we first write it down in the form

$$w = w(y,t) = \frac{MR^2}{\mu\alpha^2}\left[\left(\frac{1}{i}\cos D + \sin D\right) - y^{-1/2} e^{E}\left\{\frac{1}{i}\cos D\cos E + \sin D\cos E + \cos D\sin E + i\sin D\sin E\right\}\right]$$

The real part of the fluid velocity is thus

$$w = w(y,t) = \frac{MR^2}{\mu\alpha^2}\left[\sin D - y^{-1/2} e^{E}\left\{\sin D\cos E + \cos D\sin E\right\}\right]$$

$$= \frac{MR^2}{\mu\alpha^2}\left[\sin D - y^{-1/2} e^{E}\left\{\sin\left(D+E\right)\right\}\right]$$

$$= \frac{MR^2}{\mu\alpha^2}\left[\sin(nt-\varphi) - y^{-1/2} e^{-\frac{\alpha}{\sqrt{2}}(1-y)}\left\{\sin\left[nt-\varphi-\frac{\alpha}{2}(1-y)\right]\right\}\right]$$

THE MODIFIED FORM OF THE SOLUTION OF THE EQUATION DEFINING THE OSCILLATORY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A STRAIGHT, RIGID, CIRCULAR TUBE

We consider equation 2-18, describing the flow velocity of an oscillatory viscous fluid in a straight, rigid, circular tube:

$$w = w(y,t) = \frac{AR^2}{i\mu\alpha^2}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right]e^{int} \qquad (2\text{-}18)$$

This equation will provide velocity profiles as a function of A, R, μ, α and r. In its present form the equation is difficult for calculation purposes. We will therefore modify it in order to obtain an expression that may be easily calculated, and then demonstrate some of the velocity profiles. Accordingly, we express the Bessel functions appearing in equation 2-18 in modulus and phase form as follows.

$$M_0(\alpha) = \left|J_0(i^{3/2}\alpha)\right|$$

$$M_0(\alpha y) = \left|J_0(i^{3/2}\alpha y)\right|$$

$$\theta_0(\alpha) = \text{Phase}\left\{J_0(i^{3/2}\alpha)\right\}$$

$$\theta_0(\alpha y) = \text{Phase}\left\{J_0(i^{3/2}\alpha y)\right\}$$

Thus, equation 2-18 may be written as

$$w = w(y,t) = \frac{AR^2}{i\mu\alpha^2}\left\{1 - \frac{M_0(\alpha y)\,e^{i\theta_0(\alpha y)}}{M_0(\alpha)\,e^{i\theta_0(\alpha)}}\right\}e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2}\left\{1 - \frac{M_0(\alpha y)}{M_0(\alpha)}\,e^{i[\theta_0(\alpha y) - \theta_0(\alpha)]}\right\}e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2}\left\{1 - h_0\,e^{-i\delta_0}\right\}e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2}\left\{1 - h_0\cos\delta_0 + i h_0 \sin\delta_0\right\}e^{int}$$

(2-31)

where $h_0 = \dfrac{M_0(\alpha y)}{M_0(\alpha)}$ and $\delta_0 = \theta_0(\alpha) - \theta_0(\alpha y)$

Equation 2-31 may be simplified according to the following scheme. We set

$$M_o' = \left[(h_o \sin\delta_o)^2 + (1 - h_o \cos\delta_o)^2 \right]^{1/2}$$

$$= \left[1 + h_o^2 - 2h_o \cos\delta_o \right]^{1/2}$$

$$\tan\varepsilon_o = \frac{h_o \sin\delta_o}{1 - h_o \cos\delta_o} .$$ See figure 3.

Moreover

$$M_o' \cos\varepsilon_o = 1 - h_o \cos\delta_o$$

$$M_o' \sin\varepsilon_o = h_o \sin\delta_o$$

$$M_o \sin\varepsilon_o = h_o \sin\delta_o$$

Thus

$$M_o' \cos\varepsilon_o + i M_o' \sin\varepsilon_o = 1 - h_o \cos\delta_o + i h_o \sin\delta_o \tag{2-32}$$

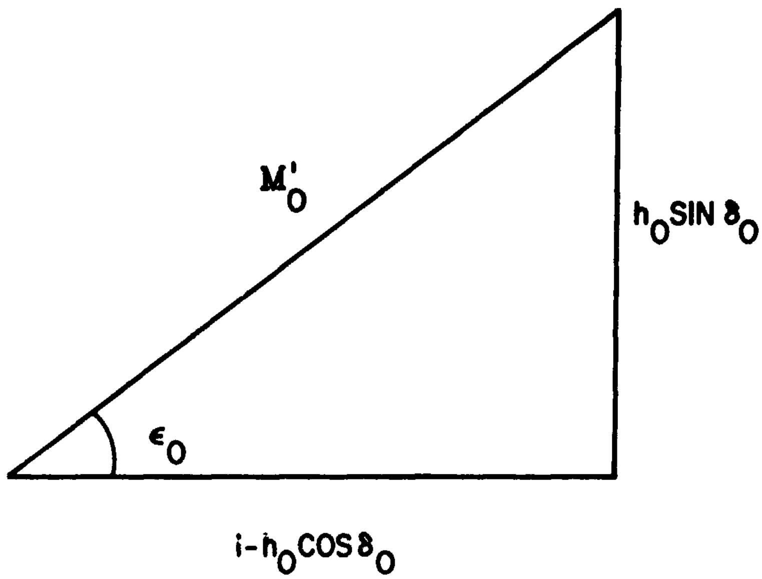


Figure 3. Diagram Illustrating the Modulus and Phase Form of Equation 2-18

Combining equations 2-31 and 2-32, we obtain

$$w = w(y,t) = \frac{AR^2}{i\mu\alpha^2}\left\{M_o' \cos\varepsilon_o + i\,M_o' \sin\varepsilon_o\right\} e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2}\left\{M_o' e^{i\varepsilon_o}\right\} e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2} M_o' e^{i(nt+\varepsilon_o)} \qquad (2\text{-}33)$$

If there is a phase lag of the flow rate behind the pressure gradient, and the latter is of the form given by equation 2-20 instead of the form given by equation 2-5, then we replace the factor Ae^{int} appearing in equation 2-33 by the factor $Me^{i(nt - \phi)}$. Thus from equation 2-33 we have

$$
\begin{aligned}
w = w(y,t) &= \frac{MR^2}{i\mu\alpha^2} M_o' e^{i(nt-\varphi+\varepsilon_o)} \\
&= \frac{MR^2}{i\mu\alpha^2} M_o' \left[\cos(nt-\varphi+\varepsilon_o) + i\sin(nt-\varphi+\varepsilon_o)\right] \\
&= \frac{MR^2}{\mu\alpha^2} M_o' \left[\frac{1}{i}\cos(nt-\varphi+\varepsilon_o) + \sin(nt-\varphi+\varepsilon_o)\right]
\end{aligned}
\tag{2-34}
$$

The real part of equation 2-34 describes the actual flow velocity along the tube axis. Thus

$$
w = w(y,t) = \frac{MR^2}{\mu\alpha^2} M_o' \sin(nt-\varphi+\varepsilon_o) \tag{2-35}
$$

In equation 2-35, note that the factor α^2 occurs in the denominator and is a factor contributing to the amplitude of the flow velocity. Clearly, as the value of α^2 increases, the amplitude of the fluid velocity decreases, i.e., the velocity profile tends to flatten out. See figure 4. Note that

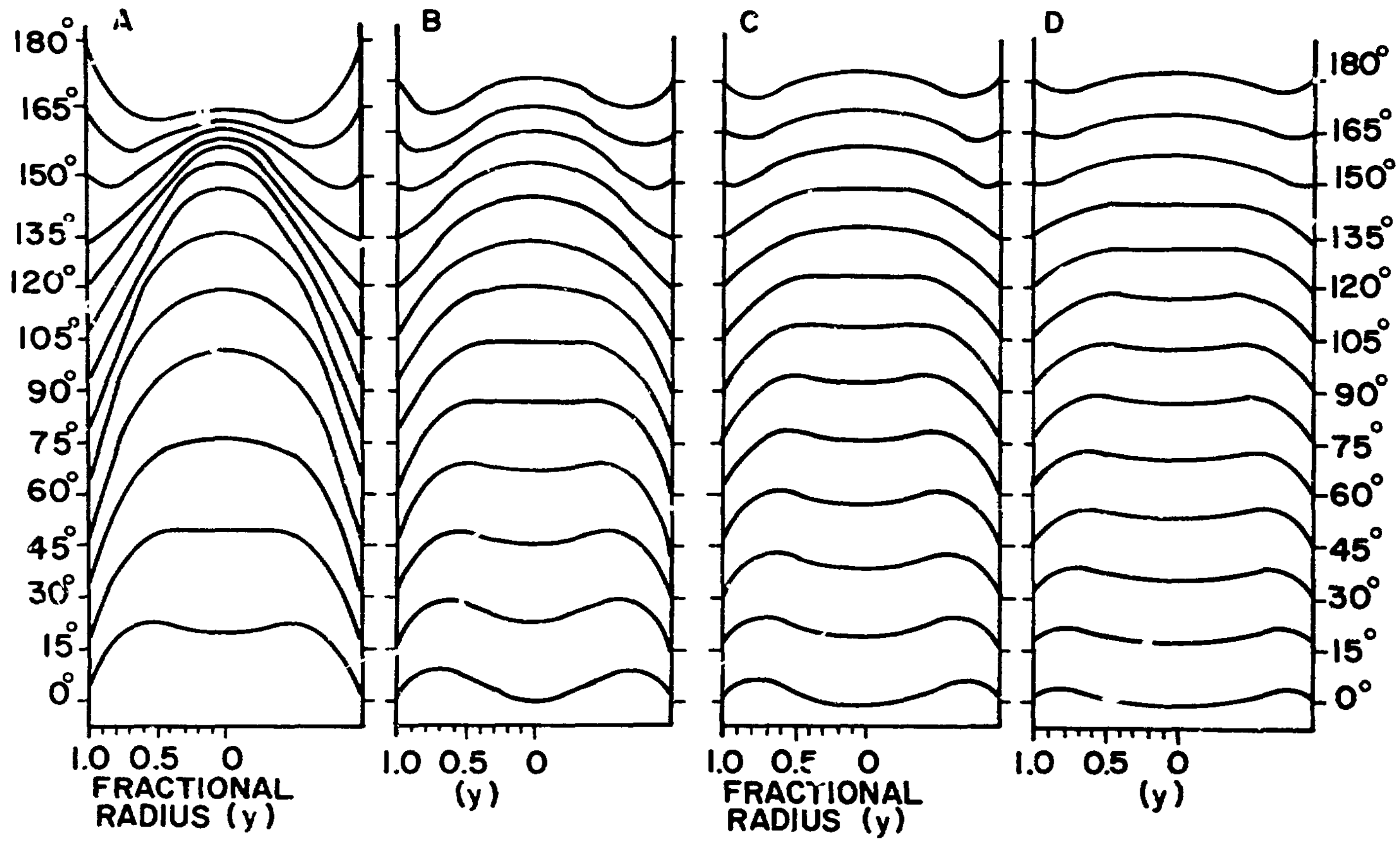


Figure 4. Velocity profiles, at intervals of 15°, of the flow arising from a sinusoidal pressure gradient in a straight, rigid tube. A, α = 3.34; B, α = 4.47; C, α = 5.78; D, α = 6.67.

since $\alpha^2 = \frac{R^2 n}{\nu}$, an increase in the value of α^2 is brought about by an increase in the values of R and/or n. Moreover, an increase in the value of α^2 is also brought about by a decrease in the value of ν.

THE VOLUME RATE OF FLOW

The volume rate of flow, Q, is obtained by integrating the fluid velocity, $w = w(y,t)$, with respect to the cross-sectional area, S, of the tube. See figure 5.

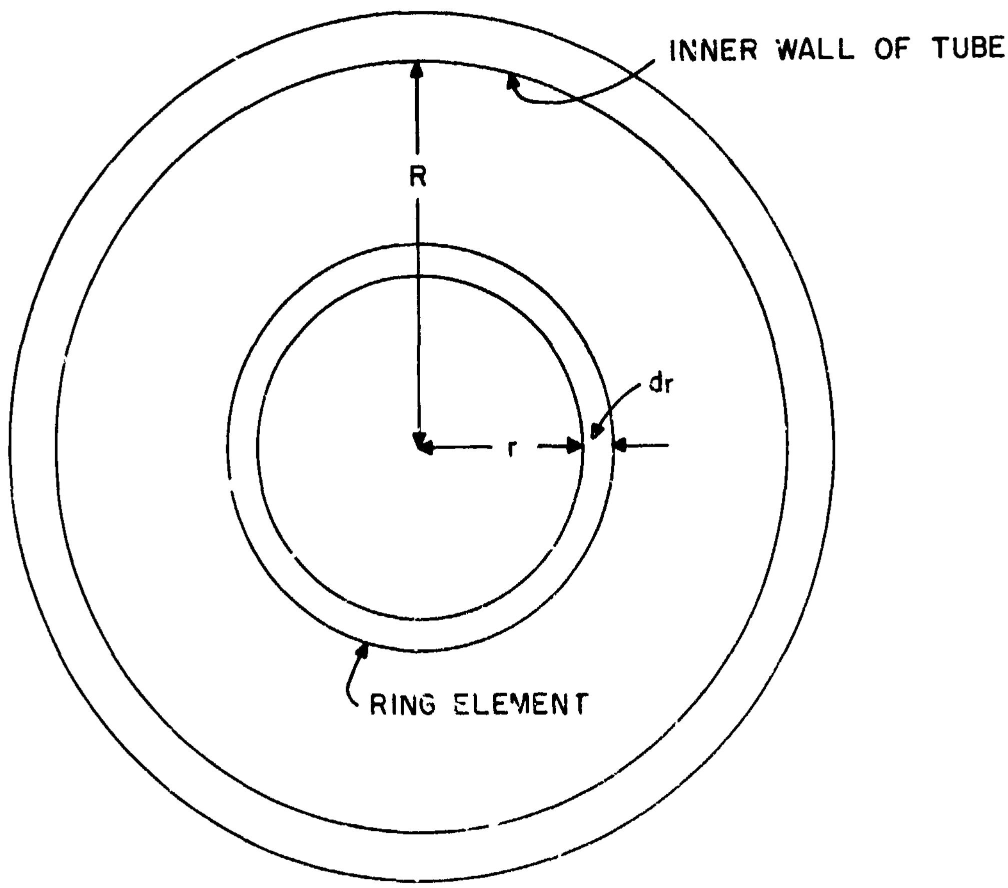


Figure 5. Ring Element of Fluid Area dS for Computing the Volume Rate of Flow

$$Q = \int_S w(y,t)\, dS$$

$$= \int_{r=0}^{r=R} w(y,t)\, 2\pi r\, dr$$

$$= \int_{y=0}^{y=1} w(y,t)\, 2\pi y R\, d(yR)$$

$$= 2\pi R^2 \int_{y=0}^{y=1} w(y,t)\, y\, dy \qquad (2\text{-}36)$$

Substituting the expression for w(y,t) from equation 2-19 into equation 2-36, we obtain

$$Q = Q(t) = \pi R^2 \left[\frac{A}{in\rho} \left\{ 1 - \frac{2\, J_1\left(i^{3/2}\alpha\right)}{i^{3/2}\alpha\, J_0\left(i^{3/2}\alpha\right)} \right\} e^{int} \right] \qquad (2\text{-}37)$$

Next, we obtain the average fluid velocity, $\bar{w} = \bar{w}(t)$, according to the relation

$$\bar{w} = \bar{w}(t) = \frac{Q(t)}{\pi R^2}$$

where Q(t) is given by equation 2-37 and πR^2 is the cross section of the tube.

Thus

$$\bar{w} = \bar{w}(t) = \frac{A}{in\rho}\left\{1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)}\right\} e^{int} \tag{2-38}$$

$$= \frac{AR^2}{i\mu\alpha^2}\left\{1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)}\right\} e^{int} \tag{2-39}$$

We may write equation 2-39 in modulus and phase form according to the following notation.

$$M_0(\alpha) = \left| J_0(i^{3/2}\alpha) \right| , \quad \theta_0(\alpha) = \text{Phase}\left\{ J_0(i^{3/2}\alpha) \right\}$$

$$M_1(\alpha) = \left| J_1(i^{3/2}\alpha) \right| , \quad \theta_1(\alpha) = \text{Phase}\left\{ J_1(i^{3/2}\alpha) \right\}$$

$$M'_{10}(\alpha) = \left| 1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)} \right|$$

$$h_{10}(\alpha) = \frac{2\,M_1(\alpha)}{\alpha\,M_0(\alpha)}$$

$$\delta_{10}(\alpha) = 135^\circ - \theta_1(\alpha) + \theta_0(\alpha) = \text{Phase}\left(i^{3/2}\right) - \theta_1(\alpha) + \theta_0$$

$$\varepsilon_{10}'(\alpha) = \text{Phase}\left\{ 1 - \frac{2\,J_1\left(i^{3/2}\alpha\right)}{i^{3/2}\alpha\,J_0\left(i^{3/2}\alpha\right)} \right\} = \text{Phase}\left\{ 1 - h_{10}(\alpha)\,e^{-i\delta_{1}} \right.$$

See figures 6 and 7.

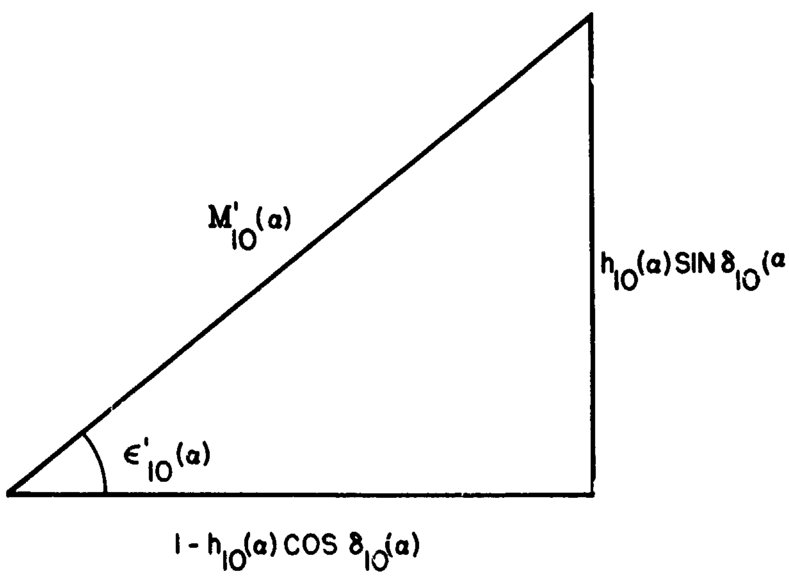


Figure 6. Diagram Illustrating the Modulus and Phase Form of Equation 2-39

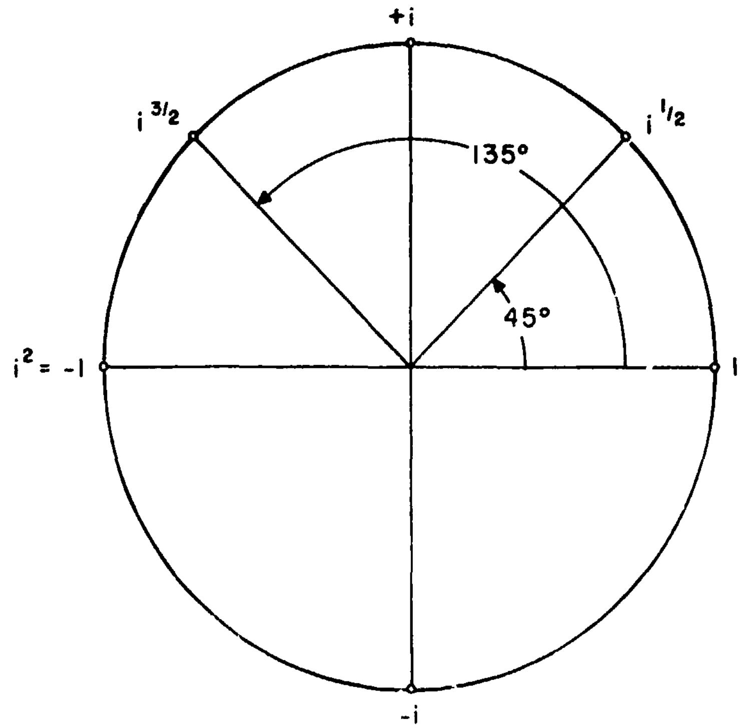


Figure 7. Diagram Illustrating the Complex Quantity $i^{3/2}$

Thus, equation 2-39 may be written as

$$\bar{w} = \bar{w}(t) = \frac{AR^2}{i\mu\alpha^2}\left\{ M'_{10}(\alpha)\, e^{i\varepsilon'_{10}(\alpha)} \right\} e^{int}$$

$$= \frac{AR^2}{i\mu\alpha^2} M'_{10}(\alpha)\, e^{i\left[nt + \varepsilon'_{10}(\alpha)\right]} \qquad (2\text{-}40)$$

If there is a negative phase lag ϕ between the flow velocity and the applied pressure gradient and the latter is of the form given by equation 2-20, instead of the earlier form described by equation 2-5, then we replace the factor Ae^{int} in equation 2-40 by the factor $Me^{i(nt - \phi)}$. Thus, from equation 2-40 we have

$$\bar{w} = \bar{w}(t) = \frac{MR^2}{i\mu d^2} M'_{10}(\alpha)\, e^{i\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right]}$$

$$= \frac{MR^2}{i\mu d^2} M'_{10}(\alpha)\left\{\cos\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right] + i \sin\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right]\right\}$$

$$= \frac{MR^2}{\mu d^2} M'_{10}(\alpha)\left\{\frac{1}{i}\cos\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right] + \sin\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right]\right\}$$

(2-41)

The actual average flow velocity along the tube axis is described by the real part of equation 2-41:

$$\bar{w} = \bar{w}(t) = \frac{MR^2}{\mu d^2} M'_{10} \sin\left[nt - \varphi + \varepsilon'_{10}(\alpha)\right] \qquad (2\text{-}42)$$

The actual volume rate of flow, Q, corresponding to the actual average flow velocity given by equation 2-42, is

$$Q = Q(t) = \bar{w}(t)\{\text{cross-sectional area of tube}\}$$

$$= \bar{w}(t)\{\pi R^2\}$$

$$= \frac{M\pi R^4}{\mu\alpha^2} M'_{10}(\alpha) \sin\left[nt - \phi + \varepsilon'_{10}(\alpha)\right] \qquad (2\text{-}43)$$

The values of the quantities M'_{10}, M'_{10}/α^2 and ε'_{10} are given in tables I, II and III of (Womersley, 1957), for values of α ranging from $\alpha = 0$ to $\alpha = 10$ at intervals of 0.05 in α. Womersley's tables have been extended by van Brummeln, 1961).

In order to calculate w(t) and Q(t) for values of the parameter α greater than 10, we may use the asymptotic expansions (McLachlan, 1961) of the modulus $M'_{10}(\alpha)$ and the phase $\varepsilon'_{10}(\alpha)$:

$$M'_{10}(\alpha) = 1 - \frac{\sqrt{2}}{\alpha} + \frac{1}{\alpha^2}$$

$$\varepsilon'_{10}(\alpha) = \frac{\sqrt{2}}{\alpha} + \frac{1}{\alpha^2} + \frac{19}{24\sqrt{2}\,\alpha^3}$$

From equation 2-43 we note that since the maximum value of $\sin\left[nt - \phi + \varepsilon'_{10}(\alpha)\right]$ is 1, we may write

$$Q_{max}(t) = |Q(t)| = \frac{M\pi R^4}{\mu\alpha^2} M'_{10} \qquad (2\text{-}44)$$

Moreover, we note that the volume rate of flow under steady, laminar conditions, according to Poiseuille's formula, is

$$Q = Q_{steady} = \frac{\pi R^4}{8\mu L}(p_1 - p_2) \qquad (2\text{-}45)$$

For a comparison of $Q_{max}(t)$ with Q_{steady}, we take the ratio of equations 2-44 and 2-45 and obtain

$$\frac{Q(t)_{MAX}}{Q_{STEADY}} = \frac{\frac{M\pi R^4}{\mu\alpha^2} M'_{10}(\alpha)}{\frac{\pi R^4}{8\mu L}(p_1 - p_2)} = \frac{8MM'_{10}(\alpha)}{\alpha^2\left(\frac{p_1 - p_2}{L}\right)} \tag{2-46}$$

In equation 2-46, if we set the magnitude of the oscillatory pressure gradient, M, equal to the pressure gradient, $\frac{p_1 - p_2}{L}$, in Poiseuille flow, we obtain

$$\frac{Q_{max}(t)}{Q_{steady}} = \frac{8M'_{10}(\alpha)}{\alpha^2} \tag{2-47}$$

In equation 2-47, note that the ratio $\frac{Q_{max}(t)}{Q_{steady}}$ decreases as α increases.

For the variation of M'_{10} and ε'_{10} as a function of α (i.e., as a function of frequency, n, since $\alpha^2 = R^2 n/\nu$), we plot the ratio $8M'_{10}/\alpha^2$ against α. See figure 2-8. According to equation 2-47, this figure also indicates the variation of the ratio

$$\frac{Q_{max}\text{: maximum flow due to a given harmonic pressure gradient}}{Q_{steady}\text{: Poiseuille flow corresponding to the given pressure gradient}}$$

with α.

From figure 8, note that as $\alpha \to 0$, $\frac{Q_{max}}{Q_{steady}} \to 1$, i.e., for smaller and smaller values of the frequency of oscillation, the maximum flow due to a given harmonic pressure gradient may be approximated by Poiseuille's formula. For values of α greater than 1, the maximum flow due to a given harmonic pressure gradient decreases rapidly as compared with the corresponding Poiseuille flow. At $\alpha = 10$, $Q_{max} = (\frac{1}{15})Q_{steady}$.

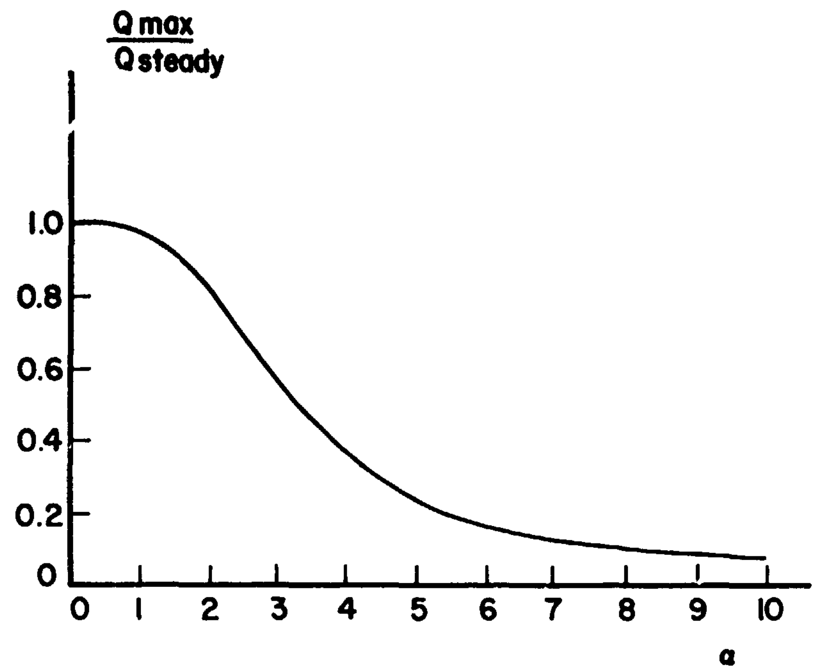


Figure 8. The variation of the ratio $\frac{Q_{max}}{Q_{steady}}$ with respect to α, assuming laminar flow where the driving pressure is of the same magnitude as the pressure gradient.

This wide variation of Q_{max} with respect to α, raises the question: How much is the value of α likely to vary in different animals? If we work with the following information:

The driving pressure is harmonic of frequency, $n = 2\pi f$;

$2R$ = diameter of the human femoral artery = 0.5 cm;

f = pulse rate = 72 per minute;

ν = kinematic viscosity of blood = 0.038 stoke;

then the value of α is obtained as

$$\alpha = R\left(\frac{n}{\nu}\right)^{1/2}$$

$$= \left(\frac{0.5}{2}\right)\left(\frac{2\pi \times 72}{60} \times \frac{1}{0.038}\right)^{1/2}$$

$$= 3.52$$

The corresponding values of α for the rabbit and the cat are of about the same magnitude. This indicates similarity in arterial flow in all these animals, and shows that the oscillating flow in the great arteries in these experimental animals and in man has the same form, and differs only in scale.

Figure 9 shows the variation of the phase lag (i.e., of $90° - \varepsilon'_{10}$) between the oscillating fluid pressure and the corresponding volume rate of flow with respect to the frequency of oscillation. Note that

$$\varepsilon'_{10}(\alpha) = \text{Phase}\left\{1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)}\right\}$$

The graph shows that the phase lag decreases with increasing frequency, and approaches its asymptotic value of 90° for large values of α.

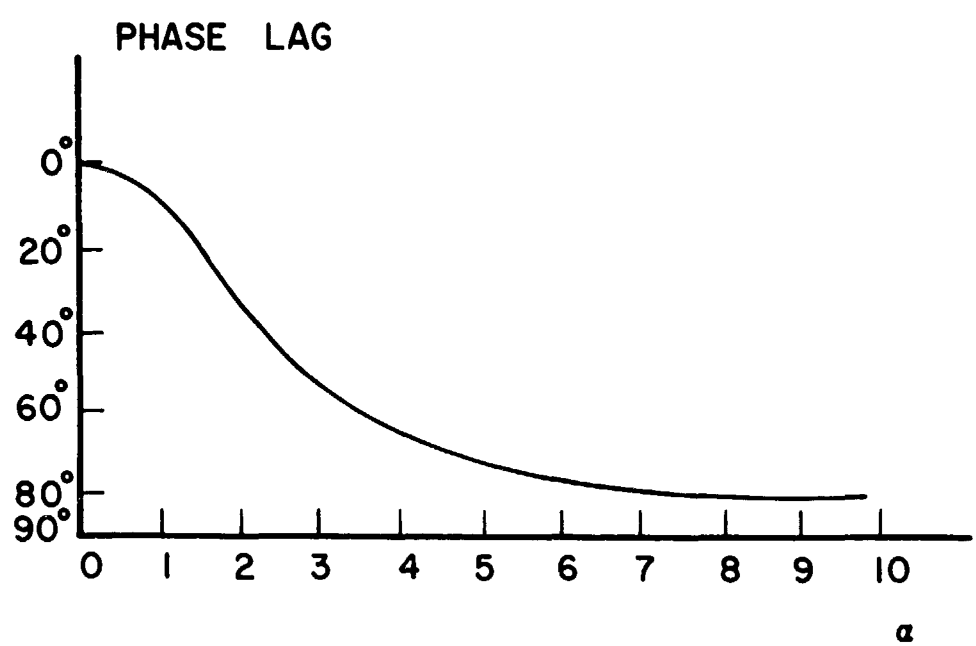


Figure 9. The variation of phase lag between the oscillating fluid pressure and the corresponding volume rate of flow as a function of α.

ELECTRICAL ANALOGUES OF FLOW QUANTITIES

It is convenient to consider the arterial circulation in a state of oscillatory motion analogous to an electrical circuit. In accordance with electrical terminology, if we associate steady flow with the "DC theory" of electricity, we may associate oscillatory fluid flow with the "AC theory" of electricity. Thus we may make the following analogies

1. Oscillating fluid pressure gradient analogous with voltage drop.
2. Oscillating fluid velocity (or volume rate of flow) analogous with electric current.
3. Fluid friction per length of tube section encountered by the fluid flow through the tube length analogous with electrical resistance.
4. Elasticity of the tube wall analogous with electrical capacitance.

Consider a fluid flowing in a rigid, frictionless tube. See figure 2-10. According to Newton's law

$$\text{force} = (\text{mass})(\text{acceleration})$$

i.e.

$$(p_1 - p_2)A = m\left(\frac{d\bar{w}}{dt}\right)$$

or

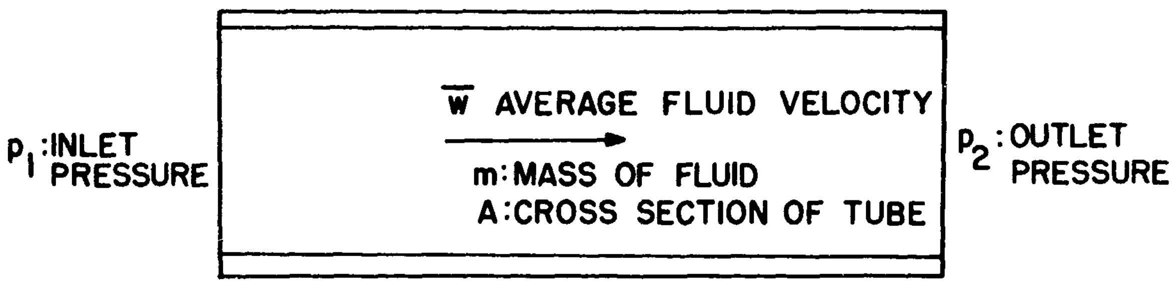
$$(p_1 - p_2)A = m\,\frac{d}{dt}\left(\frac{Q}{A}\right)$$

and

$$p_1 - p_2 = \frac{m}{A^2}\,\frac{dQ}{dt}$$

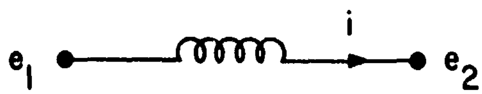
Thus, if the fluid pressure gradient $(p_1 - p_2)$ and the volume rate of flow, Q, are respectively analogous to the voltage drop, $(e_1 - e_2)$, and the current, i, then the quantity, m/A^2, of the fluid system is analogous to the inductance, L, of the electrical system. It follows that the model for a rigid, frictionless tube is an inductor. If we include fluid friction in the fluid system, then the model becomes a resistance-inductance series arrangement. Moreover, if we consider the tube in the fluid system to be flexible, then the model becomes a resistance-inductance-capacitance series arrangement. Such a model, although it represents a first approximation to the actual physiological system, provides some insight in regard to the parameters that govern the operation of the actual system.

FLUID SYSTEM



$$(p_1 - p_2)A = m\frac{d\overline{w}}{dt}$$

ELECTRICAL SYSTEM



$$e_1 - e_2 = L\frac{di}{dt}$$

Figure 10. Electrical Analogue of Rigid, Frictionless Tube

In the treatment of AC circuits, we have to consider the phase difference between the applied voltage and the current flowing in the circuit. Analogously, we may regard the oscillating arterial pressure gradient generating a flow with a phase lag. Moreover, in electrical circuits, the complex impedance Z_{elec} is the ratio of the voltage, V, impressed on the circuit and the current, I, in the circuit,

$$Z_{elec} = V/I$$

By analogy, we define the fluid impedance Z_{fluid} as

$$Z_{fluid} = \frac{\text{fluid pressure gradient}}{\text{average fluid velocity}}$$

We have noted earlier that the representations for the pressure gradient and the average fluid velocity are respectively

$$-\frac{\partial p}{\partial z} = Me^{i(nt - \phi)} \tag{2-20}$$

$$\bar{w}(t) = \frac{MR^2}{i\mu\alpha^2} M'_{10}(\alpha)e^{i[nt - \phi + \varepsilon'_{10}(\alpha)]} \tag{2-41}$$

Thus

$$Z_{fluid} = \frac{Me^{i(nt-\varphi)}}{\frac{MR^2}{i\mu\alpha^2} M'_{10}(\alpha)\, e^{i[nt - \varphi + \varepsilon'_{10}(\alpha)]}}$$

$$= \frac{i\mu\alpha^2}{R^2 M'_{10}(\alpha)}\, e^{-i\varepsilon'_{10}(\alpha)}$$

$$= \frac{i\mu\alpha^2}{R^2 M'_{10}(\alpha)} \left[\cos\varepsilon'_{10}(\alpha) - i\sin\varepsilon'_{10}(\alpha)\right]$$

$$Z_{FLUID} = \frac{\mu \alpha^2}{R^2 M_{10}'(\alpha)} \left[i \cos \varepsilon_{10}'(\alpha) + \sin \varepsilon_{10}'(\alpha) \right]$$

(2-48)

Moreover, we may write

$$Z_{elec} = R_{elec} + iX_{elec} = R_{elec} + i2\pi f L_{elec} \quad (2\text{-}49)$$

where R_{elec}, X_{elec} and L_{elec} are respectively the resistance, the reactance and the inductance of the electrical circuit. Comparing the right-hand sides of equations 2-48 and 2-49, we conclude that

$$\text{fluid resistance} = \frac{\mu \alpha^2}{R^2 M_{10}'(\alpha)} \sin \varepsilon_{10}(\alpha)$$

$$\text{fluid reactance} = \frac{\mu \alpha^2}{R^2 M_{10}'(\alpha)} \cos \varepsilon_{10}'(\alpha)$$

$$\text{fluid inductance} = \left(\frac{1}{2\pi f}\right)\left(\frac{\mu \alpha^2}{R^2 M_{10}'(\alpha)}\right) \cos \varepsilon_{10}'(\alpha)$$

$$= \frac{f}{M_{10}'(\alpha)} \cos \varepsilon_{10}'(\alpha)$$

In coastrast with electrical circuits (where the resistance and inductance are independent of the frequency of the system), we note that both the fluid resistance and inductance are functions of the frequency of oscillation of the system. The variation of fluid resistance and inductance n a rigid tube, with respect to frequency, are shown in figures 11 and 12.

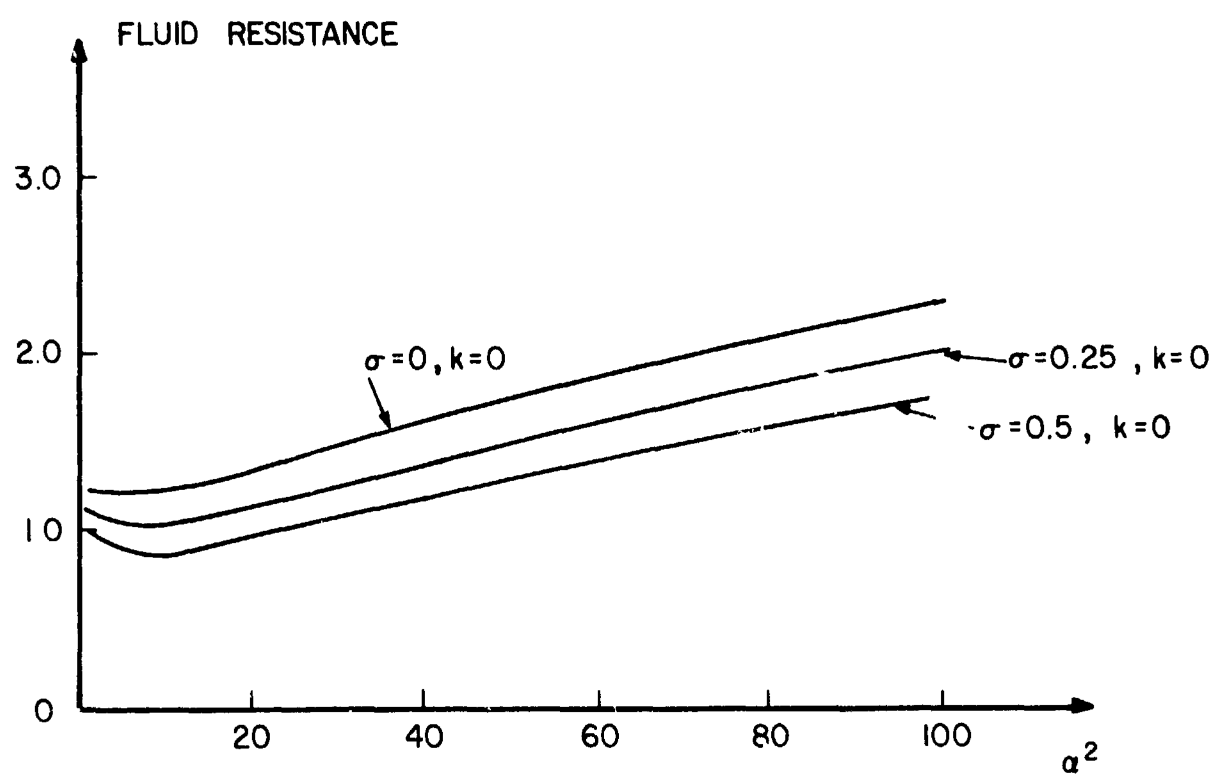


Figure 11. Variation of Fluid Resistance with α^2

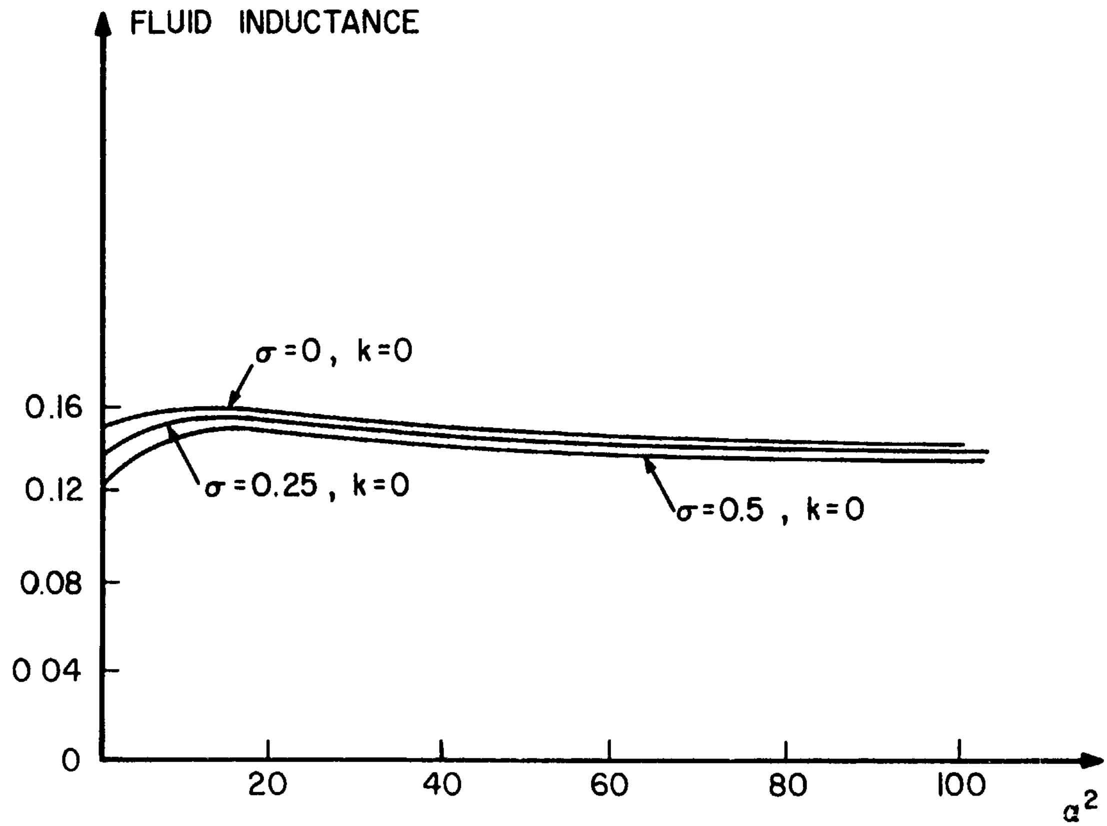


Figure 12. Variation of Fluid Inductance with α^2

Note the linear variation of fluid resistance with frequency for values of $\alpha > 4$. The variation of fluid inductance with frequency is small. For large values of the frequency, the fluid inductance remains essentially constant. The variation of fluid impedance with frequency is shown in figure 13.

Clearly, a more complete electrical analogue for the arterial circulation in a state of oscillatory motion must also include a capacitance to allow for the elasticity of the tube wall.

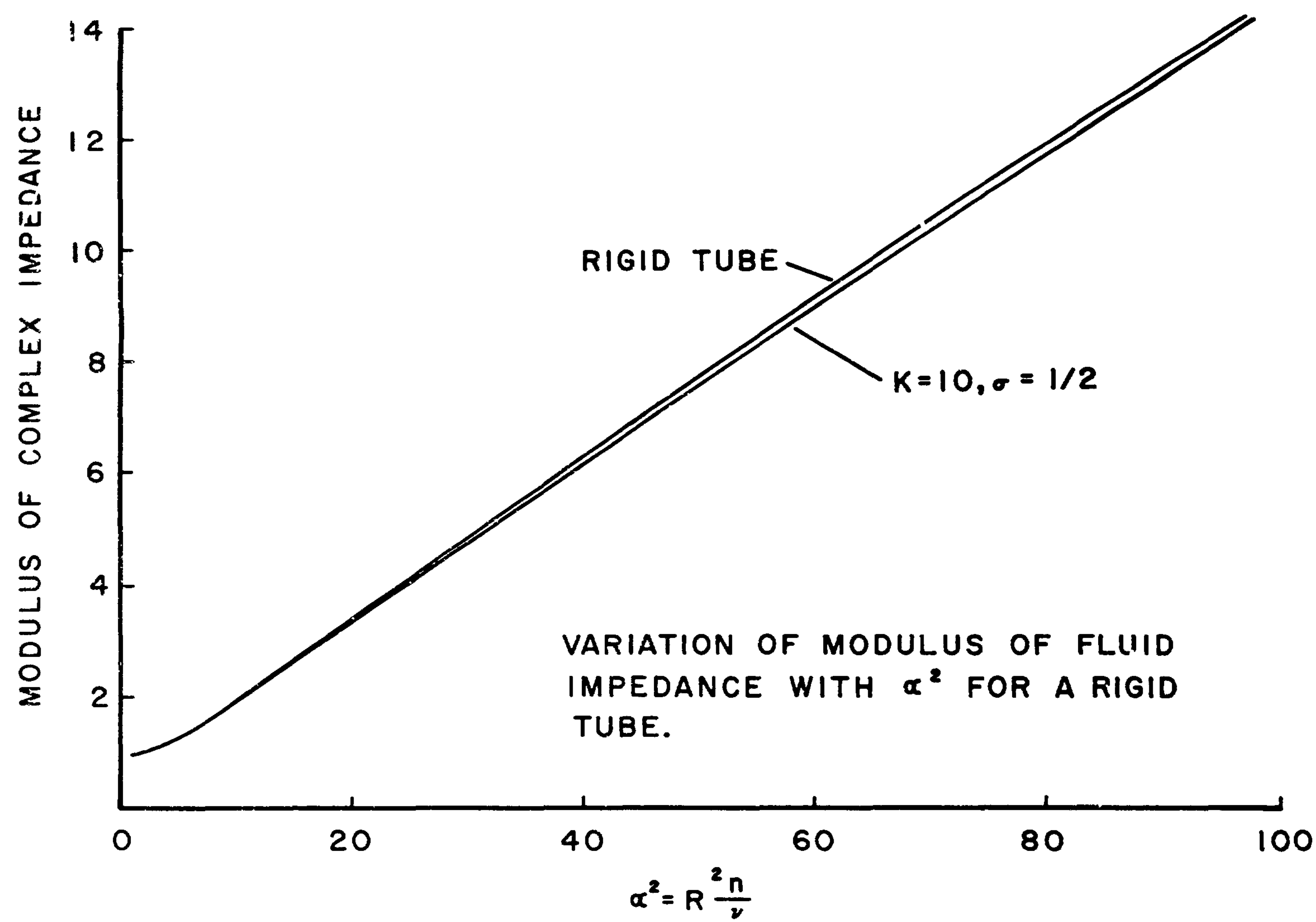


Figure 13. Variation of the Modulus of Fluid Impedance with Respect to α^2 for a Rigid Tube

FOURIER SERIES REPRESENTATION FOR CALCULATING THE VOLUME RATE OF FLOW WHEN THE PRESSURE GRADIENT IS MEASURABLE

Consider a function of time, $F(t)$, which has an oscillatory frequency n. We may write this function in the form

$$F(t) = A_o + A_1 \cos nt + A_2 \cos 2nt + \ldots + A_m \cos mnt + B_1 \sin nt + B_2 \sin 2nt + \ldots + B_m \sin mnt \qquad (2\text{-}50)$$

where A_o denotes the mean value of $F(t)$. Note that each harmonic component of the function $F(t)$ is represented by a pair of terms of the form $A_m \cos mnt + B_m \sin mnt$. Equation 2-50 may be written more conveniently in the form

$$F(t) = A_o + \sum_m (A_m \cos mnt + B_m \sin mnt)$$

If $F(t)$ represents the applied periodic pressure gradient which has magnitude M_m for the m^{th} harmonic and ϕ_m represents the negative phase lag between the fluid velocity and the applied pressure gradient for the m^{th} harmonic, then we may represent $F(t)$ in the form

$$F(t) = A_o + \sum_m M_m \cos (mnt - \phi_m)$$

where

$$M_m = \left[A_m^2 + B_m^2 \right]^{1/2}$$

$$\phi_m = \tan^{-1} \frac{B_m}{A_m}$$

In view of $\alpha^2 = R^2n/\nu \quad R^2n\rho/\mu$, equation 2-43 may be written as

$$Q(t) = \frac{M\pi R^4}{\mu} \frac{\mu}{R^2 n\rho} M'_{10}(\alpha) \sin [nt - \phi + \varepsilon'_{10}(\alpha)]$$

$$= (\pi R^2) \frac{M}{n\rho} M'_{10}(\alpha) \sin [nt - \phi + \varepsilon'_{10}(\alpha)] \qquad (2\text{-}51)$$

From equation 2-51, we may write down the contribution to the volume rate of flow made by the m^{th} harmonic in the form

$$Q_m = Q_m(t) = (\pi R^2)(M'_m/mn\rho)M_{10}(\alpha_m)\ \sin\left[mnt - \phi_m + \varepsilon'_{10}(\alpha_m)\right] \tag{2-52}$$

In equation 2-52, α_m is the value of the flow parameter α corresponding to the m^{th} harmonic, i.e., $\alpha_m^2 = m\alpha_1{}^2$ where α_1 is the value of α corresponding to the pulse frequency.

Rearranging and expanding equation 2-52, we have

$$Q_m = Q_m(t) = (\pi R^2/mn\rho)M_m M'_{10}(\alpha_m)\ \sin\left\{\left[mnt + \varepsilon'_{10}(\alpha_m)\right] - \phi_m\right\}$$

which may be written as

$$Q_m = (\pi R^2/mn\rho)\ \sin mnt\ \left\{A_m\left[M'_{10}(\alpha_m)\ \cos \varepsilon'_{10}(\alpha_m)\right] + B_m\left[M'_{10}(\alpha_m)\ \sin \varepsilon'_{10}(\alpha_m)\right]\right\}$$

$$+ (\pi R^2/mn\rho)\ \cos mnt \left\{A_m\left[M'_{10}(\alpha_m)\ \sin \varepsilon'_{10}(\alpha_m)\right] - B_m\left[M'_{10}(\alpha_m)\ \cos \varepsilon'_{10}(\alpha_m)\right]\right\} \tag{2-53}$$

In equation 2-37, for computing the volume rate of oscillatory flow, we have to evaluate the factor

$$1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)}$$

If we are concerned with the m^{th} harmonic, we have to evaluate a corresponding term of the form

$$1 - \frac{2J_1(i^{3/2}\alpha_m)}{i^{3/2}\alpha_m\, J_0(i^{3/2}\alpha_m)}$$

According to our notation, we write this as

$$1 - \frac{2J_1(i^{3/2}\alpha_m)}{i^{3/2}\alpha_m\, J_0(i^{3/2}\alpha_m)} = M'_{10}(\alpha_m)\cos\varepsilon'_{10}(\alpha_m) + i\,M'_{10}(\alpha_m)\sin\varepsilon'_{10}(\alpha_m) \tag{2-54}$$

We note that the terms $M'_{10}(\alpha_m)\cos\varepsilon'_{10}(\alpha_m)$ and $M'_{10}(\alpha_m)\sin\varepsilon'_{10}(\alpha_m)$ on the right-hand side of equation 2-54 also appear on the right-hand side of equation 2-53. Therefore, for calculating the volume rate of flow from the pressure gradient, we refer to a table of the real and imaginary parts of the factor

$$1-\frac{2J_1(i^{3/2}\alpha_m)}{i^{3/2}\alpha_m J_0(i^{3/2}\alpha_m)}$$

For this purpose we use the abbreviations

$$\begin{aligned} C_m &= \left\{1-\frac{2J_1(i^{3/2}\alpha_m)}{i^{3/2}\alpha_m J_0(i^{3/2}\alpha_m)}\right\}_{\text{REAL}} \\ &= 1-\frac{2M_1}{\alpha M_0}\cos\delta_{10} \\ &= M'_{10}(\alpha_m)\cos\varepsilon'_{10}(\alpha_m) \end{aligned} \tag{2-55}$$

$$\begin{aligned} D_m &= \left\{1-\frac{2J_1(i^{3/2}\alpha_m)}{i^{3/2}\alpha_m J_0(i^{3/2}\alpha_m)}\right\}_{\text{IMAGINARY}} \\ &= \frac{2M_1}{\alpha M_0}\sin\delta_{10} \\ &= M'_{10}(\alpha_m)\sin\varepsilon'_{10}(\alpha_m) \end{aligned} \tag{2-56}$$

Substituting the values of C_m and D_m from equations 2-55 and 2-56 into equation 2-53, we obtain the contribution to the volume rate of flow due to the m^{th} harmonic in the form

$$Q_m = Q_m(t) = (\pi R^2/mn\rho)\left[A_m C_m + B_m D_m\right] \sin mnt$$
$$+ (\pi R^2/mn\rho)\left[A_m D_m - B_m C_m\right] \cos mnt \qquad (2\text{-}57)$$

Equation 2-57 may be used for calculating the volume rate of flow, Q_m, when the pressure gradient is known in the form $M_m\ e^{i(mnt - \phi_m)}$. The values of the quantities C_m and D_m are given in table 4 of Womersley (1957) for values of α ranging from $\alpha = 0$ to $\alpha = 10$ at intervals of 0.05 in α.

We have obtained equation 2-57 without considering any perceptible reflection of the pulse wave. If such reflections are present, then this equation is not valid for calculating the volume rate of flow. The effect of reflections is considered in section VII, "Junctions and Discontinuities."

McDonald (1955) has made measurements of pressure gradient, figure 14, and volume rate of flow in the femoral artery of the dog. The volume rate of flow was obtained from the average fluid velocity across the tube, which was measured by following the motion of a gas bubble in the artery by means of high speed cinematography. A comparison of the observed volume rate of flow with that calculated from equation 2-52 is shown in figure 15. The pulse frequency was 3 cycles per second. The assumed values of the other pertinent quantities were as follows:

Radius of artery - 0.15 cm

Viscosity of blood - 0.04 poise

Density of blood - 1.05 gm/cc

Using $\alpha^2 = \frac{R^2 n}{\nu}$, $\alpha = 3.34$ for the fundamental. From figure 15, note that the agreement between theory and experiment is good, despite the drastic nature of the assumption used in deriving equation 2-52, namely that the artery is a rigid tube, and that the formula contains no disposable* constants.

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* α is concocted by definition according to $\alpha^2 = \frac{R^2 n}{\nu}$ and is not a physiological constant.

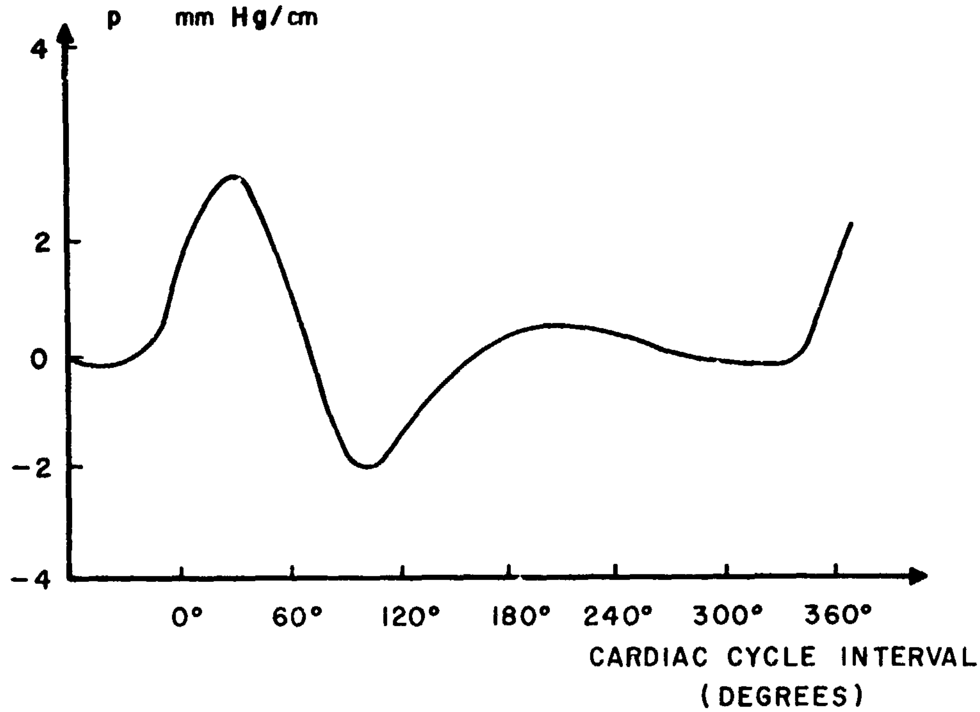

Figure 14. Observed Pressure Gradient Over One Pulse Cycle in the Femoral Artery of the Dog (McDonald, 1955)

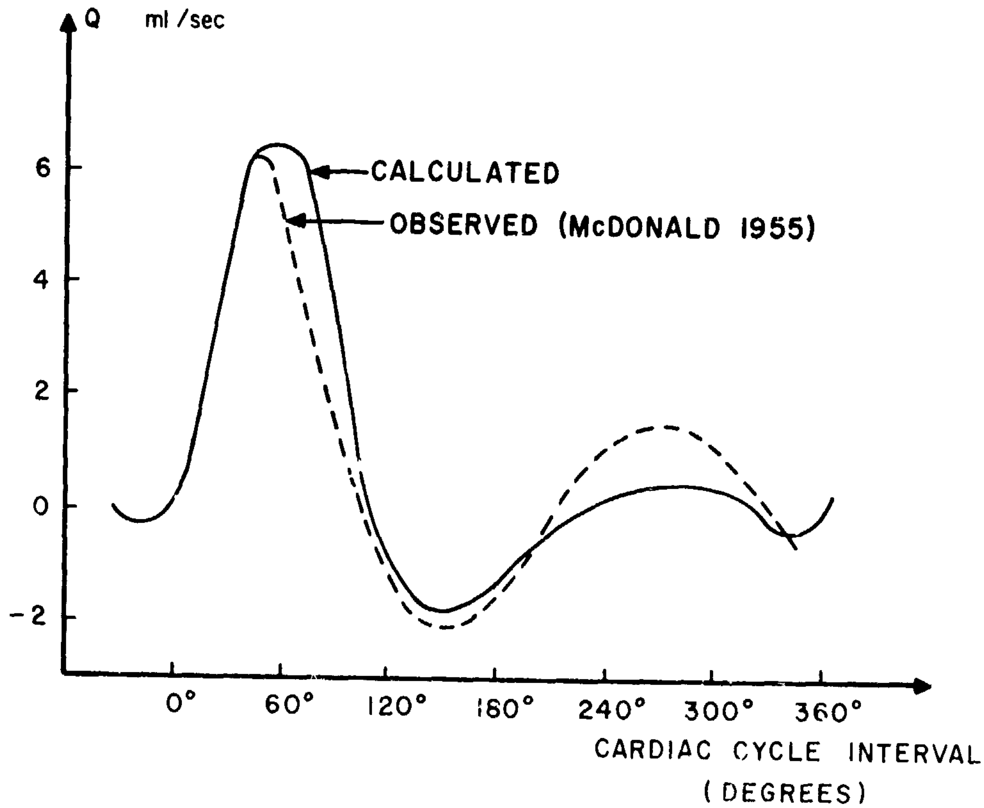

Figure 15. Volume Rate of Flow Over One Pulse Cycle in the Femoral Artery of the Dog

It will be seen in section VI, "Pressure-Flow and Pressure-Diameter Relationships," that the equations describing velocity and flow based upon the assumption of a rigid tube may be obtained from considerations of an elastic tube under limiting conditions of stiff constraint. Moreover, good agreement existing between the rigid-tube equations and McDonald's (1955) results is admissible as evidence in considering whether the conditions of stiff constraint apply to the artery.

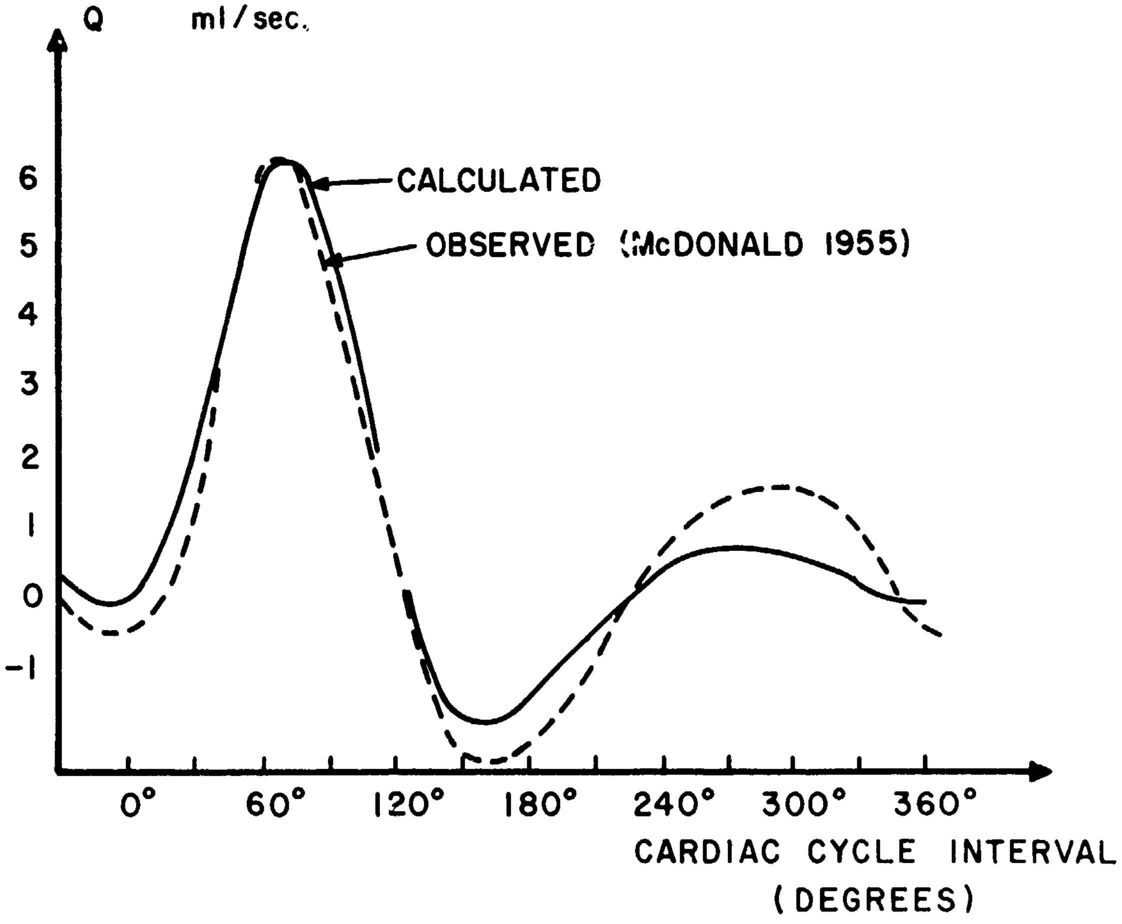

Figure 16. Comparison of Calculated and Observed Flow Over One Pulse Cycle in the Femoral Artery of the Dog

RELATIONSHIP BETWEEN THE PRESSURE GRADIENT AND THE TIME RATE OF CHANGE OF PRESSURE

In the arterial system, if we assume that the pressure gradient is generated by a periodic cardiac pulse wave having finite velocity, there is a local increase in pressure in the elastic tube. This local increase in pressure causes a local deformation in the elastic tube which is propagated along the tube like the wave of a plucked violin string traveling down the string. This phenomenon is called a pressure wave. If this pressure wave, denoted by $p(z,t)$, is considered to be harmonic in form, we may describe it by

$$p(z,t) = p_o\, e^{in(t - z/c)} \tag{2-58}$$

where c is the velocity of wave propagation and p_o is a real constant denoting the magnitude of the pressure wave.

From equation 2-58 we note the following:

1. Pressure gradient $= -\frac{\partial p}{\partial z} = -\, p_o(-\frac{in}{c})e^{in(t - z/c)}$.

2. Rate of change of pressure with respect to time $= \frac{\partial p}{\partial t}$

$$= p_o(in)e^{in(t - z/c)}.$$

Thus the pressure wave form described by equation 2-58, traveling without distortion at a velocity c, will satisfy the equation

$$-\frac{\partial p}{\partial z} = \frac{1}{c}\frac{\partial p}{\partial t} \tag{2-59}$$

Equation 2-59 has the solution $p = f_1(z - ct)$ which means any analytic function whatever of the variable $(z - ct)$. If we consider the equation

$$\frac{\partial p}{\partial z} = \frac{1}{c}\frac{\partial p}{\partial t}$$

we find that its solution has the form $p = f_2(z + ct)$ which again means any analytic function whatever of the variable $(z + ct)$.

It can be easily verified that the combined expression

$$p = f_1(z - ct) + f_2(z + ct) \tag{2-60}$$

satisfies the differential equation

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{2-61}$$

Equation 2-61 is known as the wave equation. In equation 2-60 the component solution $p = f_1(z - ct)$ is known as the propagated wave and the component solution $p = f_2(z + ct)$ is known as the reflected wave.

For a description of the propagated and reflected waves, we plot the functions $f_1(z - ct)$ and $f_2(z + ct)$ at successive values of the time, t, i.e., for t = 0, t = 1, t = 2, etc. We find that the function $f_1(z - ct)$ defines a graph of fixed form advancing forward (propagated wave) along the z-axis at the velocity c. See figure 17. Similarly, tor the function $f_2(z + ct)$, we find that the plot as a whole of unchanging form slides backward (reflected wave) along the z-axis at the velocity c. See figure 18.

The general solution $p = f_1(z - ct) + f_2(z + ct)$ implies that the function $f_1(z - ct) + f_2(z + ct)$ defines a flow pattern of general forms partly traveling forward and partly backward along the z-axis, without mutual interference and at a velocity, c, relative to the underlying fluid flow.

From equation 2-59 we note that if we know $\partial p/\partial t$ and the ressure wave velocity, c, then we can determine the pressure gradient, $\partial p/ z$, and the volume rate of flow, Q. Now, from experimental evidence, the technique required for measuring the time rate of change of pressure, $\partial p/\partial t$, is simpler than that required for measuring the pressure gradient, $\partial p/\partial z$. Thus, if the value of c is known, we may use the product of 1/c and the Fourier expansion of $\partial p/\partial t$ for calculating the volume rate of flow, Q. This procedure would imply that all the harmonic components of the pressure wave are traveling at the same velocity, c. However, the pressure wave velocity, c, is independent of the frequency only when we consider a circulatory system in which

1. the tube is perfectly elastic;
2. the fluid is inviscid;
3. the tube is so long that no reflection of the wave occurs.

Under these conditions, the pressure wave will travel without distortion.

Let the Fourier series for the flow pressure, p, have the form

$$p = p(t) = p_o + \sum_m (C_m \cos mnt + D_m \sin mnt) \tag{2-62}$$

where the right-hand side is composed of a time independent mean pressure, p_o, and a sum of osciliatory components. For determining the pressure gradient,

$\partial p/\partial z$, according to the method outlined in the preceding paragraph, we have to obtain $\partial p/\partial t$. From equation 2-62 we note that $\partial p/\partial t$ will contain only oscillatory components since p_o is a constant. Thus, the expression for $\partial p/\partial z$ will contain only oscillatory components. Therefore, according to this method of determining $\partial p/\partial z$ and the volume rate of flow, Q, we are unable to check the steady flow against the constant term in the pressure gradient.

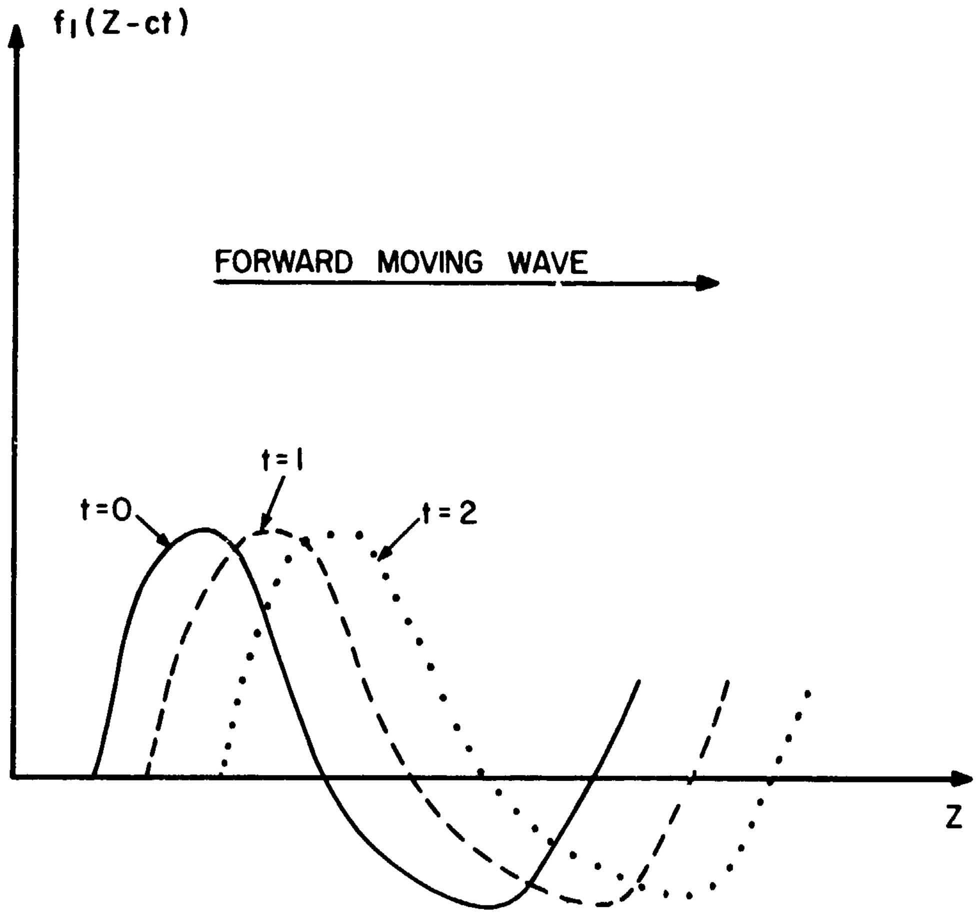

Figure 17. A pressure wave having some fixed form and moving forward with a velocity, c, relative to the underlying fluid flow. Initial position of wave at t = 0. Subsequent positions at t = 1 and t = 2.

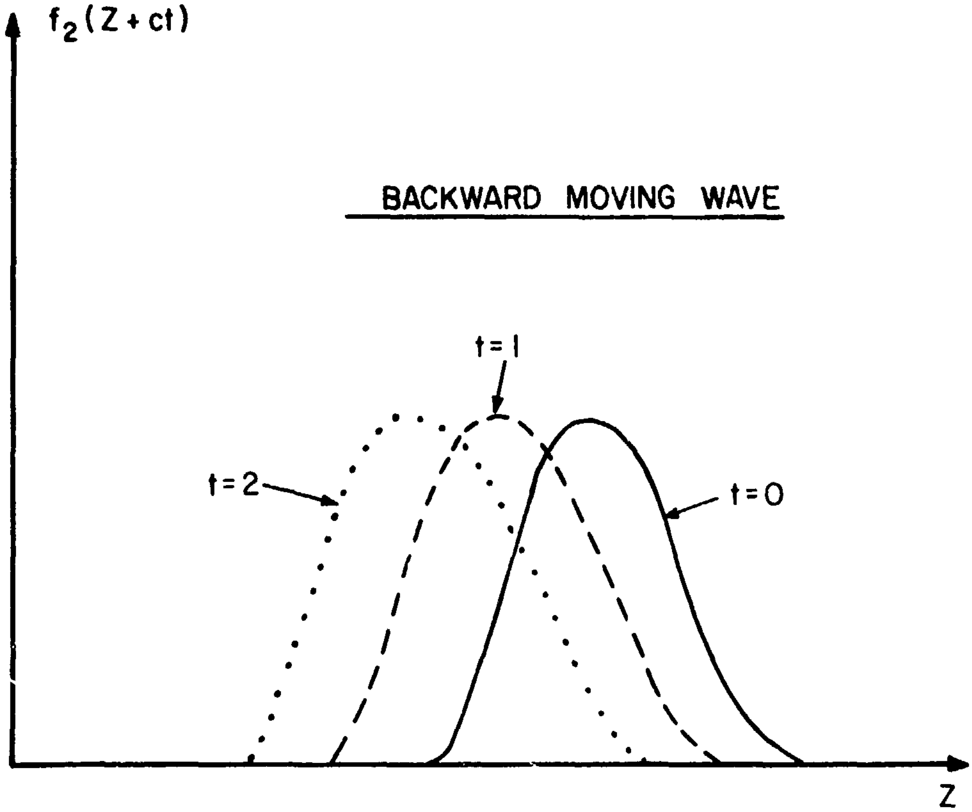

Figure 18. A pressure wave having some fixed form and moving backward with a velocity, c, relative to the underlying fluid flow. Initial position of wave at t = 0. Subsequent positions at t = 1 and t = 2.

SECTION III

MOTION OF A THIN-WALLED ELASTIC TUBE AND VELOCITY OF THE PRESSURE WAVE

INTRODUCTION

In this section we shall first examine the longitudinal and radial forces to which the tube is subjected and derive the equations of motion of the tube. Next we consider the motion of the fluid with suitable approximations and obtain expressions describing the axial and radial fluid velocity components. Finally, from the set of equations describing the motion of the tube and of the fluid, we obtain a so-called frequency equation which determines the velocity of wave propagation in terms of the parameters of the tube material and of the fluid.

THE LONGITUDINAL AND RADIAL FORCES IN THE ELASTIC TUBE

Consider an element, ABCD, of a cylindrical tube of thickness h lying between two adjacent generators, G_1G_2 and G_3G_4, of the tube and two cross sections, C_1C_2 and C_3C_4, perpendicular to the longitudinal axis of the tube. See figure 19. Let ξ, η and ζ denote the component extensions of the element of the tube along the radial, circumferential and longitudinal directions respectively.

From symmetry, the component extension, η, is zero. If the extensions ξ and ζ are considered small, then Hooke's law is applicable and we may write

$$\text{stress} = E(\text{strain})$$

where E is the linear modulus of elasticity of the tube material. Thus, along the radial direction, we have, per unit length of the tube

$$\frac{Q}{h} = E\left(\frac{\xi}{R}\right) \tag{3-1}$$

where ξ/R is the strain (change in length per unit length) along the radial direction. Equation 3-1 may be written as

$$\frac{\xi}{R} = \frac{Q}{hE} \tag{3-2}$$

This equation indicates that the dimension of Q is force per unit length.

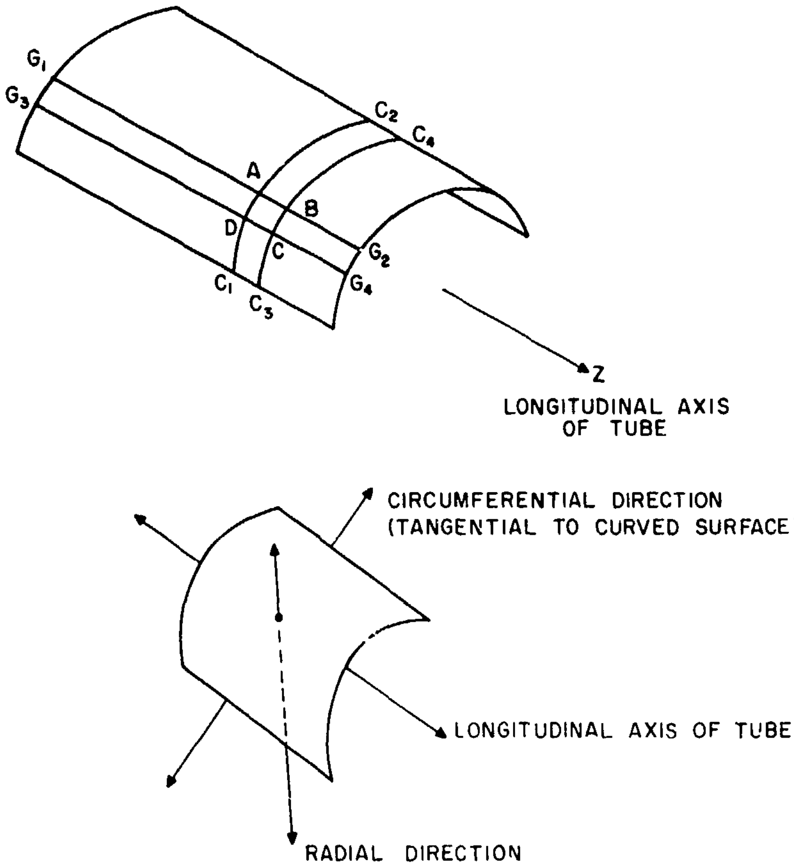

Figure 19. Forces in Cylindrical Tube.

Note that if the thickness of the tube wall is assumed to be small, then the value of the radial deformation, ξ, may be considered to remain the same at different points along the radius of the tube wall. Thus the strain along the radial direction may be written as ξ/R. However, since the value of ξ may be different at different points along the length of the tube, and $\xi = \xi(z,t)$, we have to consider $\partial\xi/\partial z$ as the value of the radial strain, ξ, at any point along the z-axis of the tube. See figure 20.

Moreover, when we consider deformation along the length of the tube, we observe that the value of the deformation, ζ, varies along the length of the tube. Since $\zeta = \zeta(z,t)$, the value of the longitudinal strain at any point along the z-axis of the tube is $\partial\zeta/\partial z$. Therefore, the relationship between stress and strain along the axis of the tube, per unit length of the tube, is

$$\frac{P}{h} = E\left(\frac{\partial\zeta}{\partial z}\right) \qquad (3\text{-}3)$$

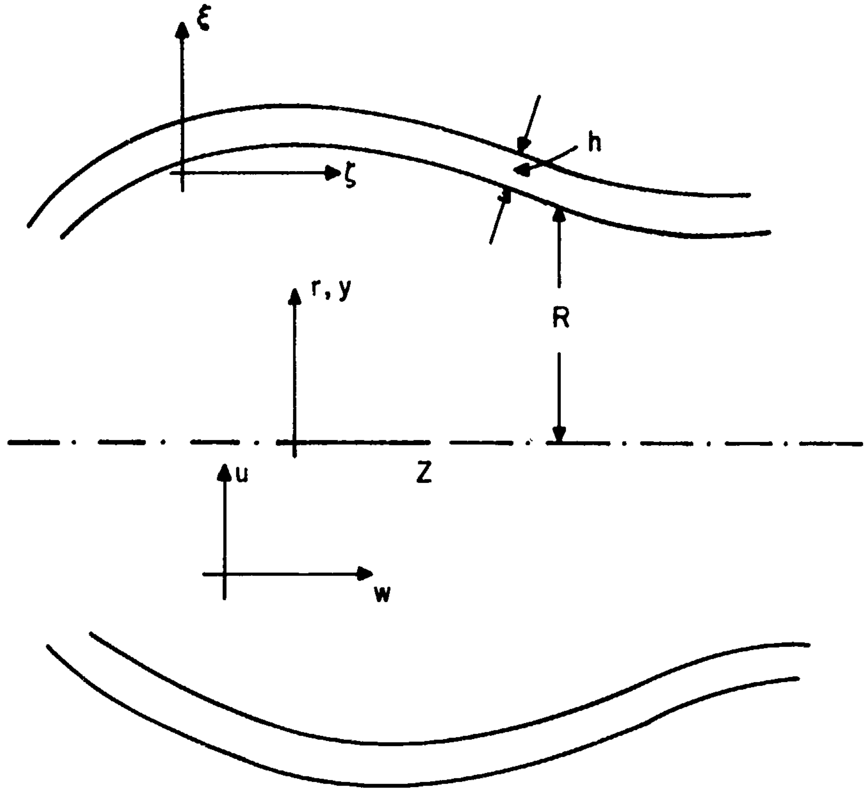

Figure 20. Elastic Tube Coordinate System.

Equation 3-3 may be written as

$$\frac{\partial \zeta}{\partial z} = \frac{P}{hE} \tag{3-4}$$

Equation 3-4 indicates that the dimension of P is force per unit length.

It is known from experiment that when an element of material is stretched in the direction of its length, it undergoes a contraction along its transverse section. The ratio

$$\frac{\text{change in length per unit length of a transverse section of material}}{\text{change in length per unit length of material}}$$

is constant within the limitations of Hooke's law. This ratio is known as Poisson's ratio and is denoted by σ.

Clearly, the longitudinal stress, P/h, causes a strain or contraction along the radial direction. From the definition of Poisson's ratio, the factor of proportionality between this longitudinal stress and radial contraction is σ. Thus we may write

$$\text{contraction along radial direction} = \sigma(P/hE) \tag{3-5}$$

Similarly, the radial stress, Q/h, causes a contraction along the longitudinal direction and we may write

$$\text{contraction along longitudinal direction} = \sigma(Q/hE) \tag{2-6}$$

Thus, the total relative change in length along the radial direction, taking contraction into account, is, from equations 3-2 and 3-5

$$\frac{\xi}{R} = \frac{Q}{hE} - \frac{\sigma P}{hE} \tag{3-7}$$

Similarly, the total relative change in length along the longitudinal axis of the tube, taking contraction into account, is, from equations 3-4 and 3-6

$$\frac{\partial \zeta}{\partial z} = \frac{P}{hE} - \frac{\sigma Q}{hE} \tag{3-8}$$

Solving equation 3-8 for P, we find

$$P = hE\,\frac{\partial \zeta}{\partial z} + \sigma Q \tag{3-9}$$

Substituting the value of Q from equation 3-7 into equation 3-9, we obtain

$$P = hE\,\frac{\partial \zeta}{\partial z} + \sigma hE\,\frac{\xi}{R} + \sigma^2 P$$

i.e.

$$P(1 - \sigma^2) = hE\left[\frac{\partial \zeta}{\partial z} + \sigma\,\frac{\xi}{R}\right]$$

$$= Bh(1 - \sigma^2)\left[\frac{\partial \zeta}{\partial z} + \sigma\,\frac{\xi}{R}\right]$$

or

$$P = Bh\left[\frac{\partial \zeta}{\partial z} + \sigma\,\frac{\xi}{R}\right] \tag{3-10}$$

Equation 3-10 describes the tension in the tube along the longitudinal axis.

By a similar procedure we also obtain the following equation describing the tension in the tube along the radial direction

$$Q = Bh\left[\frac{\xi}{R} + \sigma\,\frac{\partial \zeta}{\partial z}\right] \tag{3-11}$$

THE EQUATIONS OF MOTION OF THE ELASTIC TUBE

We will now determine the equations of motion of the elastic tube along both the longitudinal and radial directions. First consider the motion along the longitudinal axis of the tube. According to Newton's second law, the net force along the longitudinal direction acting on an element, dz, of tube wall = (mass of the element, dz)(acceleration, $\partial^2\zeta/\partial t^2$, along the z-axis)

$$\left[(P + \frac{\partial P}{\partial z}) - P\right](2\pi R) = \left[\rho h(dz)2\pi R\right]\frac{\partial^2\zeta}{\partial t^2}$$

or

$$\frac{\partial P}{\partial z} = \rho h \frac{\partial^2\zeta}{\partial t^2} \qquad (3\text{-}12)$$

See figure 21. Note that the net force per unit length acting on the element, dz, is $(P + \frac{\partial P}{\partial z} dz) - P$. The total length along which this force acts is $2\pi R$. The product of these two quantities is the net force acting on the element dz. The mass of the element dz = (density of tube material)(volume of element) = $\rho[h(dz)2\pi R]$.

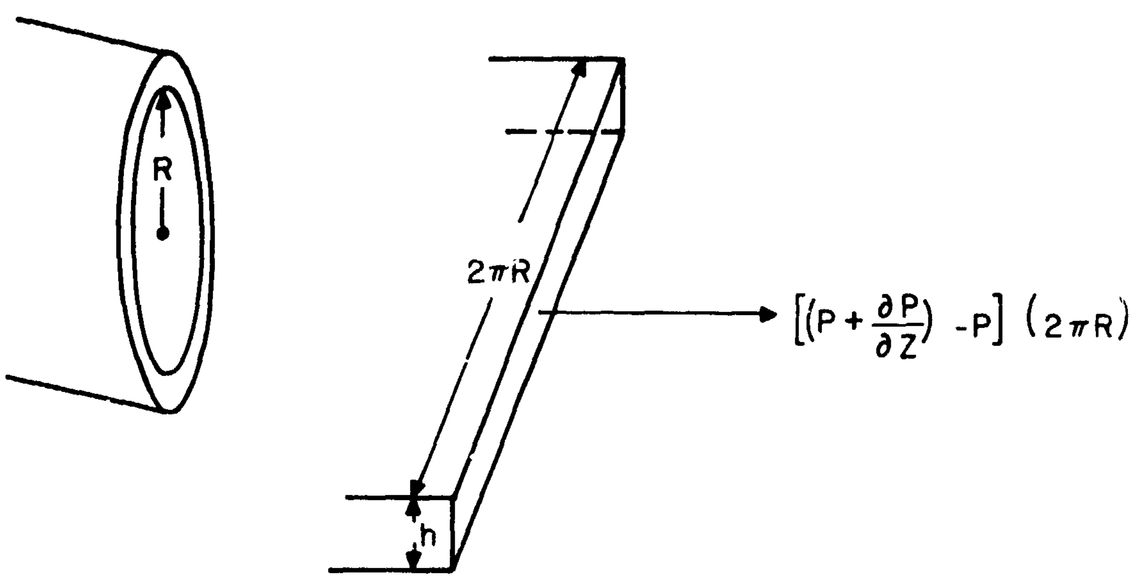

Figure 21. Forces Acting on an Element of Tube Wall.

For the motion of the tube along the radial direction we have, according to Newton's second law, the net force along the radial direction acting on an element dz = (mass of the element dz)(acceleration, $\partial^2\xi/\partial t^2$, along the radial direction)

$$p(2\pi R)(dz) - \frac{Q}{R}(2\pi R)(dz) = \rho h(dz)2\pi R\,\frac{\partial^2\xi}{\partial t^2}$$

or

$$p - \frac{Q}{R} = \rho h\,\frac{\partial^2\xi}{\partial t^2} \qquad (3\text{-}13)$$

See figure 22.

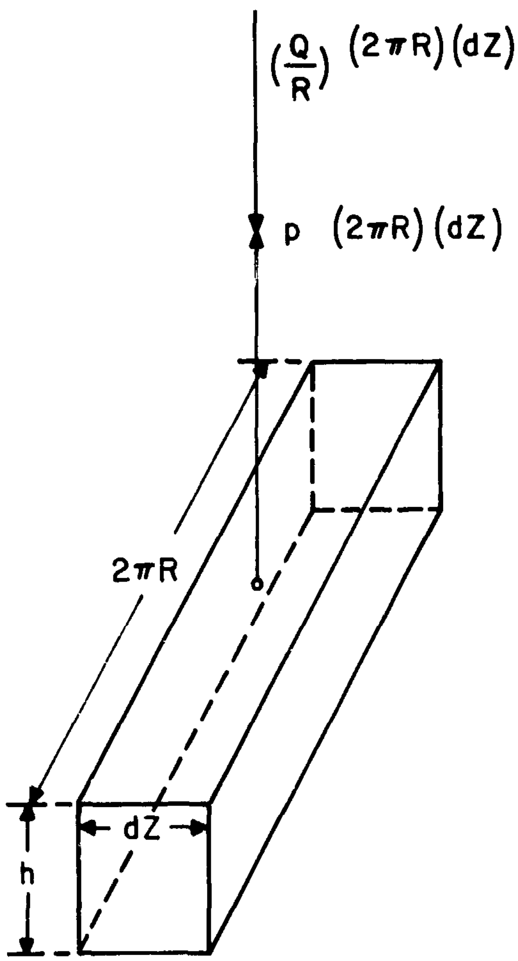

Figure 22. Forces Acting Along the Radial Direction.

If we take the viscosity of the fluid into account, there will be a surface traction on the inner surface of the tube along the longitudinal axis of the tube and equations 3-12 and 3-13 will have to be modified. The total stress due to surface traction has two components. These are

1) $\mu \left.\frac{\partial w}{\partial r}\right|_{r=R}$: radial component of the stress due to surface traction at the inner surface of the tube.

2) $\mu \left.\frac{\partial u}{\partial z}\right|_{r=R}$: longitudinal component of the stress due to surface traction at the inner surface of the tube.

Thus, the total stress due to surface traction

$$= \mu \left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right]_{r=R} \text{(area of inner surface of tube)} \tag{3-14}$$

From equation 3-10, for the tension in the tube along the longitudinal axis

$$P = Bh\left[\frac{\partial \zeta}{\partial z} + \sigma \frac{\xi}{R}\right] \tag{3-10}$$

we have, upon differentiating with respect to z:

$$\frac{\partial P}{\partial z} = Bh\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right] \tag{3-15}$$

$$\rho h \frac{\partial^2 \zeta}{\partial t^2} = Bh\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right] \tag{3-16}$$

Equation 3-16 describes the motion of the tube along the longitudinal axis, taking into account forces due to fluid pressure only. Moreover, the equation describing the motion of the tube along the radial direction, taking into account forces due to surface traction only, is

$$\rho h \frac{\partial^2 \zeta}{\partial t^2} = \mu \left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right]_{r=R} \tag{3-17}$$

Combining equations 3-16 and 3-17, we obtain equation 3-18 which describes the motion of the tube along the longitudinal axis due to the combined effects of fluid pressure and surface traction

$$\rho h \frac{\partial^2 \zeta}{\partial t^2} = \mu \left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right]_{r=R} + Bh\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right]$$

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\mu}{\rho h}\left[\frac{1}{R}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\right]_{y=1} + \frac{B}{\rho}\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right]$$

$$= \frac{\rho_0 \nu}{\rho h R}\left[\frac{\partial w}{\partial y} + R\frac{\partial u}{\partial z}\right]_{y=1} + \frac{B}{\rho}\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right] \qquad (3\text{-}18)$$

From equation 3-11 we write

$$\frac{Q}{R} = \frac{Bh}{R}\left[\frac{\xi}{R} + \sigma\frac{\partial \zeta}{\partial z}\right] \qquad (3\text{-}19)$$

Combining equations 3-13 and 3-19, we have

$$\rho h \frac{\partial^2 \xi}{\partial t^2} = p - \frac{Bh}{R}\left[\frac{\xi}{R} + \sigma\frac{\partial \zeta}{\partial z}\right]$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{p}{\rho h} - \frac{B}{\rho}\left[\frac{\xi}{R^2} + \frac{\sigma}{R}\frac{\partial \zeta}{\partial z}\right] \qquad (3\text{-}20)$$

Equation 3-20 describes the motion of the tube with respect to the radial direction.

In order to tie in the motion of the fluid and the motion of the tube, we adopt the following matching boundary conditions.

$$1) \quad u\big|_{r=R} = u\big|_{y=1} = \frac{\partial \xi}{\partial t} \qquad (3\text{-}21)$$

$$2) \quad w\big|_{r=R} = w\big|_{y=1} = \frac{\partial \zeta}{\partial t} \qquad (3\text{-}22)$$

In other words, considering that the fluid adheres to the tube wall, the values of the component fluid velocities, u and w, at the inner surface of the tube are equal to the time rate of change of the radial and longitudinal components of the tube displacements respectively.

THE EQUATIONS OF MOTION OF THE FLUID IN THE ELASTIC TUBE

The motion of the fluid in the elastic tube is governed by the equation of continuity of mass and the dynamical equations of motion along the radial and longitudinal directions.

The general form of the continuity equation in cylindrical coordinates is

$$r\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(\rho_0 ru) + \frac{\partial}{\partial\theta}(\rho_0 v) + r\frac{\partial}{\partial z}(\rho_0 w) = 0 \tag{3-23}$$

Neglecting the tangential component of the fluid velocity, u, and considering the flow to be incompressible, equation 3-23 reduces to the form

$$\frac{\partial}{\partial r}(\rho_0 ru) + r\frac{\partial}{\partial z}(\rho_0 w) = 0$$

or

$$\rho_0 r\frac{\partial u}{\partial r} + \rho_0 u + \rho_0 r\frac{\partial w}{\partial z} = 0$$

or

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

The general form of the equation of motion of the fluid along the radial direction is

$$\rho_0\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right]$$

$$= F_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} + \frac{\partial^2 u}{\partial z^2}\right] \tag{3-25}$$

If we neglect the body force, the tangential effects of the motion and the second-order effect, $\partial^2 u/\partial z^2$, equation 3-25 reduces to the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] \tag{3-26}$$

The general form of the equation of motion of the tube along the longitudinal direction is

$$\rho_0\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + w\frac{\partial w}{\partial z}\right]$$

$$= F_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial z^2}\right] \tag{3-27}$$

Neglecting the body force and the tangential effects of the fluid motion, equation 3-27 reduces to the form

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\mu}{\mu_0}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] \tag{3-28}$$

THE AXIAL AND RADIAL FLUID VELOCITY COMPONENTS IN THE ELASTIC TUBE

Suppose that we are interested in a flow process where all the variables concerned, namely, p, u and w, as functions of the independent variables r, z and t, have the following form

$$p = p(r,z,t) = p_1(r)\ e^{in(t - z/c)} \tag{3-29}$$

$$u = u(r,z,t) = u_1(r)\ e^{in(t - z/c)} \tag{3-30}$$

$$w = w(r,z,t) = w_1(r)\ e^{in(t - z/c)} \tag{3-31}$$

In these representations, p_1, u_1 and w_1 are the magnitudes of p, u and w respectively. These magnitudes are functions of the radius, r. Moreover, n is a constant denoting the frequency of the forced disturbance and c denotes the complex velocity of wave propagation.

In the above representations, equations 3-29 through 3-31, we note that

1) as time, t, increases, the argument of the function changes;
2) if the coordinate, z, increases in such a manner that the argument of the exponential function remains constant, i.e., if $t - \frac{z}{c} =$ constant, then the phases of the functions p(r,z,t), u(r,z,t) and w(r,z,t) are not altered.

Therefore, the representation of the functions described above in equations 3-29 through 3-31 is the representation of a disturbance that travels along the z-axis with a velocity c. A flow process which has the above representation is called a plane wave, since the velocity components, u and w, and the pressure, p, remain constant in any plane perpendicular to the direction of propagation, z.

Consider the relationship

$$c = f\lambda \tag{3-32}$$

where c is the velocity of wave propagation, f is the frequency of the wave and λ is the wavelength. We take the reciprocal of both sides of equation 3-32 and multiply both sides by nR. We find that

$$nR/c = nR/f\lambda = (2\pi f)R/f\lambda = 2\pi R/\lambda$$

This result indicates that if the wavelength, λ, is large compared with the inner radius of the tube, R, then the quantity nR/c is small.

From equations 3-30 and 3-31 we note that

$$u\Big|_{r=R} = u_1(R)\ e^{in(t - z/c)} \tag{3-33}$$

$$w\Big|_{r=R} = w_1(R)\ e^{in(t - z/c)} \tag{3-34}$$

Combining equations 3-33 and 3-34 with the continuity equation 3-24, we find that at r = R

$$\frac{\partial}{\partial r}\left[u_1(R)\ e^{in(t - z/c)}\right] + \frac{1}{R}\left[u_1(R)\ e^{in(t - z/c)}\right] + \frac{\partial}{\partial z}\left[w_1(R)\ e^{in(t - z/c)}\right] = 0$$

i.e. $$0 + \frac{u_1(R)}{R}\ e^{in(t - z/c)} + w_1(R)\left(-\frac{in}{c}\right)\ e^{in(t - z/c)} = 0$$

i.e. $$u_1(R)/R = w_1(R)in/c$$

i.e. $$u_1(R)/w_1(R) = inR/c \tag{3-35}$$

From equation 3-35 we observe that at the inner surface of the tube, r = R, the radial component of the fluid velocity, $u_1(R)$, as compared with the longitudinal fluid velocity, $w_1(R)$, is of order nR/c, which is small.

We will now obtain a form for the continuity equation 3-24 with the stipulation that u and w are given by equations 3-30 and 3-31. We note that from

$$u = u_1(r)\ e^{in(t - z/c)} \tag{3-30}$$

we have $$\frac{\partial u}{\partial r} = \frac{\partial u_1}{\partial r}\ e^{in(t - z/c)} \tag{3-36}$$

and $$\frac{u}{r} = \frac{u_1}{r}\ e^{in(t - z/c)} \tag{3-37}$$

Moreover, from $$w = w_1(r)\, e^{in(t - z/c)} \tag{3-31}$$

we find $$\frac{\partial w}{\partial z} = w_1(-\frac{in}{c})\, e^{in(t - z/c)} \tag{3-38}$$

Substituting these values of $\partial u/\partial r$, u/r and $\partial w/\partial z$ from equations 3-36, 3-37 and 3-38 into the continuity equation 3-24

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{3-24}$$

we find that

$$\frac{\partial u_1}{\partial r}\, e^{in(t - z/c)} + \frac{u_1}{r}\, e^{in(t - z/c)} - w_1(\frac{in}{c})\, e^{in(t - z/c)} = 0$$

or $$\frac{\partial u}{\partial r} + \frac{u_1}{r} - \frac{in}{c} w_1 = 0 \tag{3-39}$$

Since the magnitude of the radial component of the fluid velocity, u_1, is a function of r only, $u_1 = u_1(r)$, we replace the partial derivative notation with the total derivative notation and write equation 3-39 in the form

$$\frac{du_1}{r} + \frac{u_1}{r} - \frac{in}{c} w_1 = 0 \tag{3-40}$$

Moreover, since $y = r/R$ and $R(dy) = dr$, equation 3-40 has the form

$$\frac{1}{y}\frac{d}{dy}(u_1 y) = \frac{inR}{c} w_1 \tag{3-41}$$

Next, we will obtain a special form for the dynamical equation 3-26 under the condition that the fluid parameters p, u and w are as represented by equations 3-29, 3-30 and 3-31. From these representations we find that the terms $\frac{\partial u}{\partial t}$, $-\frac{1}{\rho_0}\frac{\partial p}{\partial r}$, $\nu\frac{\partial^2 u}{\partial r^2}$, $\frac{\nu}{r}\frac{\partial u}{\partial r}$ and $-\frac{\nu}{r^2}u$ of equation 3-26 may be written as

$$\frac{\partial u}{\partial t} = u_1(in)\, e^{in(t - z/c)}$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial r} = -\frac{1}{\rho_0}\frac{\partial P_1}{\partial r}\, e^{in(t - z/c)}$$

$$\nu\frac{\partial^2 u}{\partial r^2} = \nu\frac{\partial^2 u_1}{\partial r^2}\, e^{in(t - z/c)}$$

$$\frac{\nu}{r}\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial u_1}{\partial r}\, e^{in(t - z/c)}$$

$$-\frac{\nu}{r^2}\, u = -\frac{\nu}{r^2}\, u_1\, e^{in(t - z/c)}$$

Note that in equation 3-26 the terms $u\frac{\partial u}{\partial r}$ and $w\frac{\partial u}{\partial z}$ represent the inertia terms, since they have the dimension of acceleration. Moreover, we note the following

1) $$w\frac{\partial u}{\partial z} = w(-\frac{in}{c})\, e^{in(t - z/c)}$$

Thus the term $w\frac{\partial u}{\partial z}$ is of order 1/c as compared with the linear terms in equation 3-26 and may be omitted.

2) $$u\frac{\partial u}{\partial r} = u_1\, e^{in(t - z/c)}\, \frac{du_1}{dr}\, e^{in(t - z/c)}$$

$$= w_1(\frac{inR}{c})\frac{du_1}{dr}\, e^{2in(t - z/c)}$$

Thus the term $u\frac{\partial u}{\partial r}$ is of order 1/c and may also be neglected.

3) $$\frac{\partial^2 u}{\partial z^2} = (-\frac{in}{c})^2\, e^{in(t - z/c)}$$

The term $\partial^2 u/\partial z^2$ was omitted in equation 3-26, since it is or order $1/c^2$. Accordingly, equation 3-26 reduces to the form

$$u_1(in)\, e^{in(t - z/c)} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial r}\, e^{in(t - z/c)} + \nu\frac{\partial^2 u_1}{\partial r^2}\, e^{in(t - z/c)}$$

$$+\frac{\nu}{r}\frac{\partial u_1}{\partial r}\, e^{in(t - z/c)} - \frac{\nu}{r^2}\, u_1\, e^{in(t - z/c)}$$

or $$(in)u_1 = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial r} + \nu\frac{\partial^2 u_1}{\partial r^2} + \frac{\nu}{r}\frac{\partial u_1}{\partial r} - \frac{\nu}{r^2}\, u_1 \qquad (3\text{-}42)$$

Since p_1 and u_1 are functions of r only, we write equation 3-42 in the form

$$(in)u_1 = -\frac{1}{\rho_0}\frac{dp_1}{dr} + \nu\frac{d^2 u_1}{dr^2} + \frac{\nu}{r}\frac{du_1}{dr} - \frac{\nu}{r^2}\, u_1 \qquad (3\text{-}43)$$

In terms of the nondimensional parameter, $y = r/R$, note that the terms on the right-hand side of equation 3-43 may be written as

$$- \frac{1}{\rho_0} \frac{dp_1}{dr} = - \frac{1}{R\rho_0} \frac{dp_1}{dy}$$

$$\nu \frac{d^2u_1}{dr^2} = \frac{\nu}{R^2} \frac{d^2u_1}{dy^2}$$

$$\frac{\nu}{r} \frac{du_1}{dr} = \frac{\nu}{R^2y} \frac{du_1}{dy}$$

$$- \frac{\nu}{r^2} u_1 = - \frac{\nu}{R^2y^2} u_1$$

Thus equation 3-43 has the form

$$inu_1 = - \frac{1}{R\rho_0} \frac{dp_1}{dy} + \frac{\nu}{R^2} \frac{d^2u_1}{dy^2} + \frac{\nu}{R^2y} \frac{du_1}{dy} - \frac{\nu}{R^2y^2} u_1 \qquad (3\text{-}44)$$

Multiplying each term of equation 3-44 by R^2/ν, we obtain

$$\frac{d^2u_1}{dy^2} + \frac{1}{y} \frac{du_1}{dy} + i^3\alpha^2u_1 - \frac{u_1}{y^2} = \frac{R}{\mu} \frac{dp_1}{dy} \qquad (3\text{-}45)$$

Finally, we will obtain a special form for the dynamical equation 3-28 under the condition that the fluid parameters p, u and w are as described by equations 3-29, 3-30 and 3-31. We note that the terms $\frac{\partial w}{\partial t}$, $- \frac{1}{\rho_0} \frac{\partial p}{\partial z}$, $\frac{\mu}{\rho_0} \frac{\partial^2 w}{\partial r^2}$ and $\frac{\mu}{\rho_0 r} \frac{\partial w}{\partial r}$ of equation 3-28 may be written as

$$\frac{\partial w}{\partial t} = w_1 in\ e^{in(t - z/c)}$$

$$- \frac{1}{\rho_0} \frac{\partial p}{\partial z} = - \frac{1}{\rho_0} p_1 (- \frac{in}{c})\ e^{in(t - z/c)}$$

$$\frac{\mu}{\rho_0} \frac{\partial^2 w}{\partial r^2} = \nu \frac{d^2w_1}{dr^2} e^{in(t - z/c)} = \frac{\nu}{R^2} \frac{d^2w_1}{dy^2} e^{in(t - z/c)}$$

$$\frac{\mu}{\rho_0} \frac{1}{r} \frac{\partial w}{\partial r} = \frac{\nu}{r} \frac{dw_1}{dr} e^{in(t - z/c)} = \frac{\nu}{R^2y} \frac{dw_1}{dy} e^{in(t - z/c)}$$

In equation 3-28 we omit the inertia terms, $u \frac{\partial w}{\partial r}$ and $w \frac{\partial w}{\partial z}$, since they are of order $1/c$ as compared with the linear terms. Moreover, the term $\partial^2 w/\partial z^2$ is omitted, since it is of order $1/c^2$. Thus, equation 3-28 reduces to the form

$$w_1(in) = - \frac{1}{\rho_0} p_1 (\frac{in}{c}) + \frac{\nu}{R^2} \frac{d^2w_1}{dy^2} + \frac{\nu}{R^2y} \frac{dw_1}{dy} \qquad (3\text{-}46)$$

Multiplying each term of equation 3-46 by R^2/ν, we obtain

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = \frac{i^3nR^2}{c\mu} p_1 \tag{3-47}$$

We have seen in section II, equation 2-18, that the magnitude of the longitudinal fluid velocity is of the form

$$w_1 = w_1(y) = \frac{AR^2}{i\alpha^2\mu}\left[1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right] \tag{2-18}$$

Since the fluid is moving under the influence of the fluid pressure, p, we may assume the magnitude, p_1, of the pressure in the representation (3-29) to have the form

$$p_1 = A_1 J_0(ky) \tag{3-48}$$

where k is to be determined.

We will now obtain the solution of the dynamical equation (3-45). With $p_1 = A_1 J_0(ky)$, we obtain from equation 3-45:

$$\frac{d^2u_1}{dy^2} + \frac{1}{y}\frac{du_1}{dy} + (i^3\alpha^2 - \frac{1}{y^2})u_1 = \frac{R}{\mu}[- A_1 k J_1(ky)] \tag{3-49}$$

Since the function $J_1(ky)$ appears on the right-hand side of equation 3-49, we take the form of the solution as

$$u_1 = K_1 J_1(ky) \tag{3-50}$$

where K_1 is a constant to be determined. Substituting the right-hand side of equation 3-50 into the nonhomogeneous equation (3-49), we obtain

$$K_1[J_1''(ky) + \frac{1}{y} J_1'(ky) + (i^3\alpha^2 - \frac{1}{y^2})J_1(ky)] = - \frac{RA_1k}{\mu} J_1(ky) \tag{3-51}$$

Adding and subtracting $k^2J_1(ky)$ from the left hand side of equation 3-51, we have

$$K_1[J_1''(ky) + \frac{1}{y} J_1'(ky) - \frac{1}{y^2} J_1(ky) + k^2J_1(ky) - k^2J_1(ky)$$

$$+ i^3\alpha^2 J_1(ky)] = - \frac{RA_1k}{\mu} J_1(ky)$$

or $$K_1[0 - k^2 J_1(ky) + i^3\alpha^2 J_1(ky)] = -\frac{RA_1 k}{\mu} J_1(ky) \tag{3-52}$$

since $J_1(ky)$ is a solution of the corresponding homogeneous equation. Solving equation 3-52 for K_1, we obtain

$$K_1 = -\frac{RA_1 k}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right)$$

Thus the particular solution of equation 3-49 is

$$u_1 = K_1 J_1(ky) = -\frac{RA_1 k}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_1(ky) \tag{3-53}$$

The homogeneous equation corresponding to equation 3-49 is

$$\frac{d^2u_1}{dy^2} + \frac{1}{y}\frac{du_1}{dy} + \left(i^3\alpha^2 - \frac{1}{y^2}\right)u_1 = 0 \tag{3-54}$$

The solution of equation 3-54 is

$$u_1 = K_2 J_1(ky)$$

where K_2 is a constant which may be written as

$$K_2 = \frac{C_2}{J_0(i^{3/2}\alpha)}$$

by analogy with the rigid tube theory, section II. Thus the solution of the homogeneous equation 3-54 is

$$u_1 = K_2 J_1(ky) = C_2 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}$$

This is the complementary function.

The complete solution of the nonhomogeneous equation (3-49) is the sum of the two solutions, equation 3-53 and the complementary function above:

$$u_1 = C_2 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{RA_1 \hat{k}}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_1(ky) \tag{3-55}$$

Next, we determine the solution of the dynamical equation (3-47). With $p_1 = A_1 J_0(ky)$, we obtain from equation 3-47:

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = -\frac{inR^2}{c\mu} A_1 J_0(ky) \tag{3-56}$$

Since the function $J_0(ky)$ appears on the right side of equation 3-56, we take the form of the solution as

$$w_1 = K_3 J_0(ky) \tag{3-57}$$

where K_3 is a constant to be determined. Substituting the right-hand side of equation 3-57 into the nonhomogeneous equation (3-56), we obtain

$$K_3[J_0''(ky) + \frac{1}{y} J_0'(ky) + i^3\alpha^2 J_0(ky)] = -\frac{inR^2 A_1}{c\mu} J_0(ky) \tag{3-58}$$

Adding and subtracting $k^2 J_0(ky)$ from the left-hand side of equation 3-58, we have

$$K_3[J_0''(ky) + \frac{1}{y} J_0'(ky) + k^2 J_0(ky) - k^2 J_0(ky)$$

$$+ i^3\alpha^2 J_0(ky)] = -\frac{inR^2 A_1}{c\mu} J_0(ky)$$

or $$K_3[0 - k^2 J_0(ky) + i^3\alpha^2 J_0(ky)] = -\frac{inR^2 A_1}{c\mu} J_0(ky) \tag{3-59}$$

Solving equation 3-59 for K_3, we have

$$K_3 = -\frac{inR^2 A_1}{c\mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right)$$

Thus the particular solution of the nonhomogeneous equation (3-56) is

$$w_1 = K_3 J_0(ky) = -\frac{inR^2 A_1}{c\mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right) J_0(ky) \tag{3-60}$$

The homogeneous equation corresponding to equation 3-56 is

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = 0 \tag{3-61}$$

The solution of equation 3-61 is

$$w_1 = K_4 J_0(i^{3/2}\alpha y)$$

where the constant, K_4, may be written in the form

$$K_4 = \frac{C_1}{J_0(i^{3/2}\alpha)}$$

by analogy with the rigid tube theory. Thus the complementary function of equation 3-56 is

$$w_1 = K_4 J_0(i^{3/2}\alpha y) = C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \tag{3-62}$$

The complete solution of equation 3-56 is the sum of the particular integral, equation 3-60, and the complementary function, equation 3-62:

$$w_1 = C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{inR^2A_1}{c\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right)J_0(ky) \tag{3-63}$$

Now we refer to the equation of continuity

$$\frac{1}{y}\frac{d}{dy}(u_1 y) = \frac{inR}{c} w_1 \tag{3-41}$$

Note that if the representations for u_1 and w_1, obtained in equations 3-55 and 3-63 respectively, are substituted into the continuity equation (3-41), we should obtain an identity. Evaluating the right-hand side of equation 3-41,

$$\frac{inR}{c} w_1 = \frac{inRC_1}{c}\frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{i^2n^2R^3A_1}{c^2\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right)J_0(ky) \tag{3-64}$$

Evaluating the left-hand side of equation 3-41,

$$\frac{1}{y}\frac{d}{dy}(u_1 y) = \frac{du_1}{dy} + \frac{u_1}{y}$$

$$= \frac{d}{dy}\left[C_2 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{RkA_1}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_1(ky)\right]$$

$$+ \frac{1}{y}\left[C_2 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{RkA_1}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_1(ky)\right]$$

$$= C_2(i^{3/2}\alpha) \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{Rk^2A_1}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_0(ky)$$

$$+ \frac{1}{y} C_2 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{1}{y}\frac{RkA_1}{\mu}\left(\frac{1}{i^3\alpha^2 - k^2}\right) J_1(ky)$$

(3-65)

For the right-hand sides of equations 3-64 and 3-65 to be identical, we must have:

$$\frac{inR}{c} C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} = C_2(i^{3/2}\alpha) \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}$$

or

$$\frac{C_2}{C_1} = \frac{inR}{c\alpha i^{3/2}}$$

and

$$\frac{i^2 n^2 R^3 A_1}{c^2 \mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right) J_0(ky) = \frac{Rk^2 A_1}{\mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right) J_0(ky)$$

or

$$K = \frac{inR}{c}$$

This is the value of k which was to be determined. See page 67.

From the relation

$$J_0(iy) = I_0(y)$$

where I_0 is the modified Bessel function of the first kind, the assumed form of the magnitude of the pressure gradient, p_1, is

$$p_1 = A_1 J_0(ky) = A_1 J_0\left(\frac{inRy}{c}\right) = A_1 I_0\left(\frac{nRy}{c}\right) = A_1 I_0\left(\frac{nr}{c}\right)$$

Moreover, from the relation

$$J_1(iy) = iI_1(y)$$

we find, upon inserting $k = \frac{inR}{c}$, that

$$J_1(iky) = iI_1\left(\frac{inR}{c} y\right)$$

or

$$J_1(ky) = iI_1\left(\frac{nRy}{c}\right)$$

From the relation

$$k = \frac{inR}{c}$$

we note that

$$k^2 = \frac{i^2 n^2 R^2}{c^2} = -\frac{n^2 R^2}{c^2}$$

and

$$|k^2| = \frac{n^2 R^2}{c^2}$$

Clearly, the quantity (n^2R^2/c^2) is small compared with $(R^2n/\nu) = \alpha^2$. For example, for the 6^{th} harmonic of the pulse frequency of the dog, we find that the quantity

$$\frac{n^2R^2}{c^2} \cong \frac{6}{10^4}$$

and the ratio of (n^2R^2/c^2) to the corresponding value of $\alpha^2 = (R^2n/\nu)$ is about 9×10^{-3}. We are therefore justified in replacing the quantity $i^3\alpha^2 - k^2$, appearing in equations 3-55 and 3-63, by $i^3\alpha^2 = (i^3R^2n/\nu)$. Moreover, from the expansion

$$I_0(x) = 1 + \left(\frac{x}{2}\right)^2 + \frac{\left(\frac{x}{2}\right)^4}{(1^2)(2^2)} + \ldots$$

we note that for small values of x, disregarding second and higher powers of x,

$$I_0(x) = 1$$

or

$$I_0\left(\frac{nr}{c}\right) = J_0(ky) = 1$$

Similarly, from the expansion

$$I_1(x) = \frac{x}{2} + \frac{\left(\frac{x}{2}\right)^2}{(1^2)(2)} + \ldots$$

we note that for small values of x, disregarding second and higher powers of x,

$$I_1(x) = \frac{x}{2}$$

or

$$I_1\left(\frac{nRy}{c}\right) = \frac{nRy}{2c}$$

From the earlier relations

$$w = w_1(r)\ e^{in(t - z/c)} \tag{3-31}$$

$$u = u_1(r)\ e^{in(t - z/c)} \tag{3-30}$$

we note that

$$\frac{\partial^2 w}{\partial z^2} = w_1(r)\left(-\frac{in}{c}\right)^2 e^{in(t - z/c)}$$

$$\frac{\partial^2 u}{\partial z^2} = u_1(r)\left(-\frac{in}{c}\right)^2 e^{in(t - z/c)}$$

We disregarded $(\partial^2 w/\partial z^2)$ and $(\partial^2 u/\partial z^2)$ appearing in equations 3-25 and 3-28 because they are of order $1/c$. The approximations indicated in the preceding paragraph are of the same degree that is implicit in omitting the second-order terms $(\partial^2 w/\partial z^2)$ and $(\partial^2 u/\partial z^2)$ from the dynamical equations 3-25 and 3-28.

Making the approximations indicated above, namely

$$i^3\alpha^2 - k^2 \cong i^3\alpha^2 = \frac{i^3 R^2 n}{\nu}$$

$$J_0(ky) \cong 1$$

in equation 3-63, we obtain

$$w_1 = C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{inR^2 A_1}{c\mu}\left(\frac{1}{i^3 R^2 n/\nu}\right)(1)$$

$$= C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} + \frac{A_1}{\rho_0 c} \qquad (3\text{-}66)$$

Similarly, equation 3-55, with the approximations

$$i^3\alpha^2 - k^2 \cong i^3\alpha^2 = \frac{i^3 R^2 n}{\nu}$$

$$J_1(ky) = iI_1\left(\frac{nRy}{c}\right) \cong \frac{i}{2}\left(\frac{nRy}{c}\right)$$

and

$$\frac{C_2}{C_1} = \frac{inR}{i^{3/2}\alpha c}$$

assumes the form

$$u_1 = \frac{inR}{i^{3/2}\alpha c} C_1 \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{R}{\mu} \frac{inR}{c} \frac{A_1}{i^3 R^2 n/\nu} \frac{i}{2} \frac{nRy}{c}$$

$$= \frac{inR}{2c}\left[C_1 \frac{2J_1(i^{3/2}\alpha y)}{i^{3/2}\alpha J_0(i^{3/2}\alpha)} + \frac{A_1}{\rho_0 c} y\right] \tag{3-67}$$

The values of the magnitudes of the velocity components w_1 and u_1 at the inner surface of the tube, i.e., at $r = R$ or at $y = 1$, is obtained by setting $y = 1$ in equations 3-66 and 3-67. Thus

$$w_1\Big|_{y=1} = C_1 \frac{J_0(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)} + \frac{A_1}{\rho_0 c}$$

$$= C_1 + \frac{A_1}{\rho_0 c} \tag{3-68}$$

$$u_1\Big|_{y=1} = \frac{inR}{2c}\left[C_1 \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha J_0(i^{3/2}\alpha)} + \frac{A_1}{\rho_0 c}\right]$$

$$= \frac{inR}{2c}\left[C_1 F_{10}(\alpha) + \frac{A_1}{\rho_0 c}\right] \tag{3-69}$$

where $$F_{10}(\alpha) = \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha J_0(i^{3/2}\alpha)}$$

In equation 3-18, describing the motion of the tube wall, we need the value of $\left.\frac{\partial w}{\partial y}\right|_{y=1}$. From equation 3-63, we find that

$$\frac{\partial w}{\partial y} = - C_1 i^{3/2}\alpha \frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} + \frac{inR^2 A_1}{c\mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right) k J_1(ky)$$

$$\left.\frac{\partial w}{\partial y}\right]_{y=1} = - C_1 i^{3/2}\alpha \frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)} + \frac{inR^2 A_1}{c\mu} \left(\frac{1}{i^3\alpha^2 - k^2}\right) k J_1(k)$$

$$= - C_1 i^{3/2}\alpha \left[\frac{i^{3/2}\alpha}{2} F_{10}(\alpha)\right] + \frac{inR^2 A_1}{c\mu} \left(\frac{1}{i^3\alpha^2}\right)\left(\frac{inR}{c}\right) J_1\left(\frac{inR}{c}\right) \tag{3-70}$$

where we have written

$$F_{10}(\alpha) = \frac{2}{i^{3/2}\alpha} \frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}$$

$$k = \frac{inR}{c}$$

$$J_1(k) = J_1\left(\frac{inR}{c}\right)$$

$$i^3\alpha^2 - k^2 = i^3\alpha^2$$

Since $J_1\left(\frac{inR}{c}\right) = \frac{i}{2}\left(\frac{nR}{c}\right)$, equation 3-70 may be written as

$$\left.\frac{\partial w_1}{\partial y}\right|_{y=1} = - \frac{C_1}{2} i^3\alpha^2 F_{10}(\alpha) + \frac{1}{2} \frac{A_1}{\rho_0 c} \frac{n^2R^2}{c^2} \tag{3-71}$$

We will now assume that the two components of the deformation of the tube wall, ξ and ζ, have the following specific forms

$$\xi = \xi(z,t) = D_1 \, e^{in(t - z/c)} \tag{3-72}$$

$$\zeta = \zeta(z,t) = E_1 \, e^{in(t - z/c)} \tag{3-73}$$

where D_1 and E_1 are arbitrary constants. According to this description, the deformation components are harmonic and have the same frequency as the representations for p, u and w described in equations (3-29, 3-30 and 3-31.

From the deformation components of the tube wall, as described by equations 3-72 and 3-73, we shall obtain the boundary conditions for the magnitudes of the fluid velocity components u_1 and w_1. We recall the matching boundary conditions for the fluid velocity and the deformation of the tube wall

$$u = \frac{\partial \xi}{\partial t} \quad \text{at } y = 1 \tag{3-21}$$

$$w = \frac{\partial \zeta}{\partial t} \quad \text{at } y = 1 \tag{3-22}$$

From the representation

$$\zeta = \zeta(z,t) = E_1 \, e^{in(t - z/c)} \tag{3-73}$$

we have $$\frac{\partial \zeta}{\partial t} = inE_1 \, e^{in(t - z/c)}$$

and $$\left.\frac{\partial \zeta}{\partial t}\right|_{y=1} = inE_1 \, e^{in(t - z/c)} \tag{3-74}$$

Moreover, we know that at the inner surface of the tube

$$\left. w_1 \right|_{y=1} = C_1 + \frac{A_1}{\rho_0 c} \tag{3-68}$$

and $$\left. w \right|_{y=1} = w_1 \, e^{in(t - z/c)}$$

$$= \left[C_1 + \frac{A_1}{\rho_0 c}\right] e^{in(t - z/c)} \tag{3-75}$$

Combining equations 3-74 and 3-75, we have

$$inE_1 = C_1 + \frac{A_1}{\rho_0 c} \tag{3-76}$$

Similarly, from the representation

$$\xi = \xi(z,t) = D_1 \, e^{in(t - z/c)} \tag{3-72}$$

we have $$\frac{\partial \xi}{\partial t} = inD_1 \, e^{in(t - z/c)}$$

and $$\left.\frac{\partial \xi}{\partial t}\right|_{y=1} = inD_1 \, e^{in(t - z/c)} \tag{3-77}$$

Moreover, we know that at the inner surface of the tube

$$\left. u_1 \right|_{y=1} = \frac{inR}{2c}\left[C_1 F_{10}(\alpha) + \frac{A_1}{\rho_0 c}\right] \tag{3-69}$$

and $$u\Big]_{y=1} = u_1 \, e^{in(t - z/c)}\Big]_{y=1}$$

$$= \frac{inR}{2c}\left[C_1 F_{10}(\alpha) + \frac{A_1}{\rho_0 c}\right] e^{in(t - z/c)} \qquad (3\text{-}78)$$

Combining equations 3-77 and 3-78, we have

$$inD_1 = \frac{inR}{2c}\left[C_1 F_{10}(\alpha) + \frac{A_1}{\rho_0 c}\right] \qquad (3\text{-}79)$$

Equations 3-76 and 3-79 describe the boundary conditions for u_1 and w_1.

We shall now obtain the equations of motion of the tube in terms of the harmonic representations for the fluid velocity components u and w, equations 3-30 and 3-31, the fluid pressure, p, equation 3-29, and the tube wall deformation components, ξ and ζ, equations 3-72 and 3-73. We recall the equation of longitudinal motion of the tube wall in the form

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\rho_0 \nu}{\rho h R}\left[\frac{\partial w}{\partial y} + R\,\frac{\partial u}{\partial z}\right]_{y=1} + \frac{B}{\rho}\left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\,\frac{\partial \xi}{\partial z}\right] \qquad (3\text{-}18)$$

From

$$\zeta = E_1 e^{in(t - z/c)} \qquad (3\text{-}73)$$

we have

$$\frac{\partial^2 \zeta}{\partial t^2} = i^2 n^2 E_1 e^{in(t - z/c)} = - n^2 E_1 e^{in(t - z/c)}$$

$$\frac{\partial^2 \zeta}{\partial z^2} = \frac{i^2 n^2}{c^2} E_1 e^{in(t - z/c)} = - \frac{n^2 E_1}{c^2} e^{in(t - z/c)}$$

Next, from $$\xi = D_1 e^{in(t - z/c)} \qquad (3\text{-}72)$$

we have $$\frac{\partial \xi}{\partial z} = - \frac{inD_1}{c} e^{in(t - z/c)}$$

Moreover, from $$w = w(y,t,z) = w_1(y) e^{in(t - z/c)} \qquad (3\text{-}31)$$

we have $$\frac{\partial w}{\partial y} = \frac{\partial w_1}{\partial y} e^{in(t - z/c)}$$

Inserting the value of $\left.\frac{\partial w_1}{\partial y}\right|_{y=1}$ obtained in equation 3-71, we write

$$\left.\frac{\partial w}{\partial y}\right]_{y=1} = \left[-\frac{C_1}{2} i^3 \alpha^2 F_{10}(\alpha) + \frac{1}{2} \frac{n^2 R^2}{c^2} \frac{A_1}{\rho_o c} \right] e^{in(t - z/c)}$$

Finally, from

$$u = u(y,z,t) = u_1(y) e^{in(t - z/c)} \tag{3-30}$$

we have

$$\left.\frac{\partial u}{\partial z}\right]_{y=1} = \left. u_1(y)\left(-\frac{in}{c}\right) e^{in(t - z/c)} \right]_{y=1}$$

Using the value of $u_1(y)|_{y=1}$ determined in equation 3-69, we obtain

$$\left.\frac{\partial u}{\partial z}\right]_{y=1} = \frac{inR}{2c}\left[C_1 F_{10}(\alpha) + \frac{A_1}{\rho_o c} \right]\left(-\frac{in}{c}\right) e^{in(t - z/c)}$$

Since $\left.\frac{\partial u}{\partial z}\right|_{y=1}$ is of order n^2R/c^2, we neglect this term appearing in equation 3-18. Substituting the results determined above into equation 3-18, we obtain

$$-n^2 E_1 e^{in(t-z/c)} = \frac{\rho_0}{\rho}\frac{\nu}{hR} e^{in(t-z/c)} \left[-\frac{C_1}{2} i^3\alpha^2 F_{10}(\alpha) + \frac{1}{2}\frac{n^2R^2}{c^2}\frac{A_1}{\rho_0 c}\right]$$

$$+ \frac{B}{\rho}\left(-\frac{E_1 n^2}{c^2} e^{in(t-z/c)}\right) + \frac{B}{\rho}\frac{\sigma}{R}\left(-\frac{1}{c} D_1 in e^{in(t-z/c)}\right)$$

or
$$-n^2 E_1 = \frac{\rho_0}{\rho}\frac{\nu}{hR}\left[-\frac{i^3\alpha^2}{2} F_{10}(\alpha) C_1 + \frac{n^2R^2}{2\rho_0 c^3} A_1\right]$$

$$+ \frac{B}{\rho}\left[-\frac{n^2}{c^2} E_1 - \frac{i\sigma n}{Rc} D_1\right] \tag{3-80}$$

This equation is associated with the longitudinal motion of the tube wall in terms of harmonic representations for u, w, ξ and ζ.

Next, we will obtain the equation (3-20) describing the motion of the tube in the radial direction in terms of the harmonic representations for the pressure, p, equations 3-29 and 3-48, and the tube wall deformation components, ξ and ζ, equations 3-72 and 3-73. We first recall equation 3-20

$$\frac{\partial^2\xi}{\partial t^2} = \frac{p}{\rho h} - \frac{B}{\rho}\left(\frac{\xi}{R^2} + \frac{\sigma}{R}\frac{\partial\zeta}{\partial z}\right) \tag{3-20}$$

From equations 3-29 and 3-48

$$p = p_1 e^{in(t - z/c)} = A_1 J_0(ky) e^{in(t - z/c)}$$

$$= A_1 I_0\left(\frac{nR}{c}\right) e^{in(t - z/c)}$$

$$= A_1 e^{in(t - z/c)}$$

according to the approximations considered on page 73. Moreover, from

$$\zeta = \zeta(z,t) = E_1 e^{in(t - z/c)} \tag{3-73}$$

we have

$$\frac{\partial \zeta}{\partial z} = - \frac{in}{c} E_1 e^{in(t - z/c)}$$

Finally, from

$$\xi = \xi(z,t) = D_1 e^{in(t - z/c)} \tag{3-72}$$

we have

$$\frac{\partial^2 \xi}{\partial t^2} = - n^2 D_1 e^{in(t - z/c)}$$

Substituting the results determined above into equation (3-20), we obtain

$$-n^2 D_1 e^{in(t - z/c)} = \frac{p_1}{\rho h} e^{in(t - z/c)} - \frac{B}{\rho}\left[-\frac{i\sigma n}{Rc} E_1 + \frac{D_1}{R^2}\right] e^{in(t - z}$$

or

$$-n^2 D_1 = \frac{A_1}{\rho h} - \frac{B}{\rho}\left[-\frac{i\sigma n}{Rc} E_1 + \frac{D_1}{R^2}\right] \tag{3-81}$$

This equation is associated with the radial motion of the tube wall in terms of harmonic representations for p, ξ and ζ.

THE FREQUENCY EQUATION

The equations 3-76, 3-79, 3-80 and 3-81 are four homogeneous equations in the four arbitrary constants A_1, C_1, D_1 and E_1. This system of equations has solutions different from zero if the determinant of the coefficients of A_1, C_1, D_1 and E_1 is zero. By setting this determinant equal to zero, we obtain an algebraic equation (3-82) for determining the wave velocity, c, in terms of the elastic properties of the tube, the fluid parameters and the frequency, α. The algebraic equation (3-82) in terms of c or, equivalently, equation 3-83 in terms of x is called the frequency equation.

Rearranging the terms in equations 3-76, 3-79, 3-80 and 3-81 in the order A_1, C_1, D_1 and E_1, we write

$$\frac{A_1}{\rho_0 c} + C_1 + 0D_1 - inE_1 = 0 \tag{3-76}$$

$$\frac{inR}{2\rho_0 c^2} A_1 + \frac{inR}{2c} F_{10} C_1 - inD_1 + 0E_1 = 0 \tag{3-79}$$

$$\frac{A_1}{\rho h} + 0C_1 + \left(n^2 - \frac{B}{\rho R^2}\right) D_1 + \frac{i\sigma n B}{\rho R c} E_1 = 0 \tag{3-81}$$

$$\frac{\nu n^2 R}{2\rho h c^3} A_1 - \frac{i^3 \alpha^2 \rho_0 \nu F_{10}}{2\rho h R} C_1 - \frac{i\sigma n B}{\rho R c} D_1 + \left(n^2 - \frac{B n^2}{\rho c^2}\right) E_1 = 0 \tag{3-80}$$

Setting the determinant of the coefficients of A_1, C_1, D_1 and E_1 equal to zero, we have

$$\begin{vmatrix} \frac{1}{\rho_0 c} & 1 & 0 & -in \\ \frac{inR}{2\rho_0 c^2} & \frac{inRF_{10}}{2c} & -in & 0 \\ \frac{1}{\rho h} & 0 & n^2 - \frac{B}{\rho R^2} & \frac{i\sigma n B}{\rho c R} \\ \frac{\nu n^2 R}{2\rho h c^3} & \frac{-i^3 \rho_0 \nu \alpha^2 F_{10}}{2\rho h R} & \frac{-i\sigma n B}{\rho c R} & n^2\left(1 - \frac{B}{\rho c^2}\right) \end{vmatrix} = 0 \tag{3-82}$$

In the fourth row, second column we note that

$$-\frac{i^3\alpha^2\rho_0\nu}{2\rho hR}F_{10} = \frac{i\rho_0 Rn}{2\rho h}F_{10}$$

To simplify this determinant, we perform elementary operations and approximations and obtain

$$x^2[(1 - F_{10})(1 - \sigma^2)] - x[k(1 - F_{10}) + F_{10}(\frac{1}{2} - 2\sigma) + 2] + 2k + F_{10} = 0 \tag{3-83}$$

This is the so-called frequency equation in terms of the variable x.

DEDUCTIONS FROM THE FREQUENCY EQUATION

The roots of the quadratic equation (3-83) are

$$x = \frac{1}{2(1-F_{10})(1-\sigma^2)}\left\{k(1-F_{10}) + F_{10}\left(\frac{1}{2}-2\sigma\right) + 2 \pm \left\{\left[k(1-F_{10}) + F_{10}\left(\frac{1}{2}-2\sigma\right)+2\right]^2 - 4(1-F_{10})(1-\sigma^2)(2k+F_{10})\right\}^{1/2}\right\}$$

or

$$x(1-\sigma^2) = G \pm \frac{1}{2(1-F_{10})}\left\{\left[k(1-F_{10}) + F_{10}\left(\frac{1}{2}-2\sigma\right)+2\right]^2 - 4(1-F_{10})(1-\sigma^2)(2k+F_{10})\right\}^{1/2} \tag{3-84}$$

where

$$G = \frac{k}{2} + \sigma - \frac{1}{4} + \frac{1}{1-F_{10}}\left(1 - \frac{1}{4} - \sigma\right)$$

Now consider the second term on the right-hand side of equation 3-84, namely,

$$\left\{ \frac{1}{4\left(1-F_{10}\right)^2}\left[k\left(1-F_{10}\right) + F_{10}\left(\frac{1}{2} - 2\sigma\right) + 2 \right]^2 - \frac{4\left(1-F_{10}\right)\left(1-\sigma^2\right)\left(2k+F_{10}\right)}{4\left(1-F_{10}\right)^2} \right\}^{1/2}$$

This may be written as

$$\left\{ G^2 - \frac{4\left(1-F_{10}\right)\left(1-\sigma^2\right)\left(2k+F_{10}\right)}{4\left(1-F_{10}\right)^2} \right\}^{1/2}$$

or

$$\left\{ G^2 - \left(1-\sigma^2\right) H \right\}^{1/2}$$

where $H = \frac{2k + F_{10}}{1 - F_{10}} = \frac{1 + 2k - 1 + F_{10}}{1 - F_{10}} = \left(\frac{1 + 2k}{1 - F_{10}}\right) - 1$

Thus we may write equation 3-84 in the form

$$\left(1-\sigma^2\right) x = G \pm \left[\left\{ G^2 - \left(1-\sigma^2\right) H \right\} \right]^{1/2} \qquad (3\text{-}85)$$

where $$G = \frac{k}{2} + \sigma - \frac{1}{4} + \frac{\frac{5}{4} - \sigma}{1 - F_{10}} \tag{3-86}$$

and $$H = \left(\frac{1 + 2k}{1 - F_{10}} \right) - 1 \tag{3-87}$$

We recall the foll:wing notation

$$F_{10}(\alpha) = \frac{2 J_1 \left(i^{3/2} \alpha \right)}{i^{3/2} \alpha \, J_0 \left(i^{3/2} \alpha \right)}$$

$$M'_{10}(\alpha) = \left| 1 - \frac{2 J_1 \left(i^{3/2} \alpha \right)}{i^{3/2} \alpha \, J_0 \left(i^{3/2} \alpha \right)} \right|$$

$$\varepsilon'_{10}(\alpha) = \text{phase} \left\{ 1 - \frac{2 J_1 \left(i^{3/2} \alpha \right)}{i^{3/2} \alpha \, J_0 \left(i^{3/2} \alpha \right)} \right\}$$

Accordingly we write

$$1 - F_{10}(\alpha) = 1 - \frac{2 J_1 \left(i^{3/2} \alpha \right)}{i^{3/2} \alpha \, J_0 \left(i^{3/2} \alpha \right)} = M'_{10}(\alpha) \, e^{i \varepsilon'_{10}(\alpha)}$$

and $$\frac{1}{1-F_{10}(\alpha)} = \left(\frac{1}{M'_{10}(\alpha)}\right) e^{-i\varepsilon'_{10}(\alpha)} \tag{3-88}$$

Since $\varepsilon'_{10}(\alpha)$ and $M'_{10}(\alpha)$ are known, we note from equations 3-85, 3-86 and 3-87 that all the quantities for determining the roots of the frequency equation are known.

From $$F_{10}(\alpha) = \frac{2 J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)}$$

we note that $F_{10}(\alpha)$ is always complex.

Moreover, from equations 3-86 and 3-87 we note that since G and H are expressed in terms of $(1 - F_{10})$ it follows that both G and H are always complex. Finally, from equation 3-85, since x is expressed in terms of G and H, we conclude that x is always complex. Therefore, the motion of the liquid is either damped or unstable.

From equations 3-85, 3-87 and 3-88 we write

$$G = \left(\frac{5}{4} - \sigma\right)\frac{e^{-i\varepsilon_{10}(\alpha)}}{M'_{10}(\alpha)} + \left(\frac{k}{2} + \sigma - \frac{1}{4}\right) \tag{3-89}$$

$$H = (1 + 2k)\frac{e^{-i\varepsilon_{10}(\alpha)}}{M'_{10}(\alpha)} - 1 \tag{3-90}$$

We may also write equation 3-85 in the form

$$(1-\sigma^2)x = G\left\{1 \pm \left[1 - (1-\sigma^2)\frac{H}{G^2}\right]^{1/2}\right\} \qquad \text{(3-91}$$

Note that the sign of the arg x is determined by the sign of the arg G and since arg G is always negative, it follows that the motion is damped.

DAMPING OF THE PULSE WAVE

We recall the substitutions $x = kB/\rho c^2$ and $k = \rho h/R\rho_0$. From these we may write $x = (\frac{hB}{R\rho_0})\frac{1}{c^2}$ and since $B = \frac{E}{1 - \sigma^2}$, we have

$$x = \frac{hE}{R\rho_0}(\frac{1}{1 - \sigma^2})\frac{1}{c^2}$$

or

$$\frac{x}{2} = (\frac{1}{2}\frac{hE}{R\rho_0})(\frac{1}{1 - \sigma^2})\frac{1}{c^2}$$

or

$$(1 - \sigma^2)\frac{x}{2} = (\frac{1}{2}\frac{hE}{R\rho_0})\frac{1}{c^2} \qquad (3\text{-}92)$$

Now, the simplest expression for the velocity of propagation of a pressure pulse is given by the Moens-Korteweg formula

$$c_0 = (\frac{hE}{2R\rho_0})^{1/2} \qquad (3\text{-}93)$$

This formula is based upon the following assumptions:

1. The tube is thin-walled, i.e., $h << R$.
2. The fluid is incompressible, i.e., its bulk modulus is high compared with E.
3. The fluid is inviscid.

The first two assumptions above, are reasonable approximations for blood in an artery for which $h/2R < 0.1$. Moreover, the bulk modulus of water is from 10^3 to 10^4 times greater than E, the Young's modulus of the arterial wall. Regarding the third assumption above, we note that the effect of the viscosity of the fluid is great in small tubes and at low frequencies. However, in tubes comparable with the larger arteries, viscosity has the effect of reducing the predicted velocity by 5-10 per cent. This is equivalent to multiplying the right-hand side of equation 3-93 by a constant ranging in value from 0.9 to 0.95. Combining equations 3-92 and 3-93, we write

$$(1 - \sigma^2)\frac{x}{2} = \frac{c_0^2}{c^2} \qquad (3\text{-}94)$$

Here, c_0 is real and c is complex. c_0 denotes the velocity of wave propagation in an incompressible inviscid fluid enclosed in a thin-walled elastic tube and c is the complex velocity of wave propagation.

Combining equations 3-85 and 3-94, we write

$$\left(\frac{c_0}{c}\right)^2 = G \pm \left[\left\{ G^2 - (1-\sigma^2) H \right\} \right]^{1/2}$$

The solutions of this equation represent two types of waves. One solution of the frequency equation represents outgoing waves in the positive z direction. The other solution represents incoming waves. We consider only outgoing waves and therefore consider the plus sign only in the above equation.

In order to obtain c from equation 3-94, we take square roots and write c_0/c in complex form as

$$\left[(1 - \sigma^2)\frac{x}{2}\right]^{1/2} = \frac{c_0}{c} = X - iY$$

where X is the real part of c_0/c and may be considered as the wave speed parameter. Y, the imaginary part of c_0/c, may be considered as the wave damping parameter. For convenience, we may write

$$X = \text{Real } (c_0/c) = c_0 \text{ Real } (1/c) = \frac{c_0}{\dfrac{1}{\text{Real } (1/c)}} = c_0/c_1 \qquad (3\text{-}95)$$

Thus the phase velocity of the pressure wave or the measured pulse velocity, c_1, is given by

$$c_1/c_0 = 1/X$$

Since a vibrating system has its own inherent unit of time, namely its period, it is logical to refer to properties of the pressure wave "per period" (or over one wavelength) rather than "per second." One characteristic on this basis is the decay of the pressure wave over one wavelength. It can be shown that the factor exp $\{\frac{-2\pi Y}{X}\}$ determines the decay of the oscillation over one wavelength.

The decay of the disturbance or the slowing of the pulse-wave velocity must be associated with increased damping. Figure 23 shows the variation of the damping of the wave with respect to the frequency, α. Note that the damping is much greater for small values of α and is 100 per cent for $\alpha < 1$. In larger mammals, such values of damping are obtained for the fundamental wave in vessels like the saphenous artery (in the dog, $\alpha = 0.8$ to 1.0). This is the physical basis for accounting for the disappearance of the pulse wave in the arteriales even though their length is a small fraction of a wavelength.

The variation of the wave velocity ratio, c_1/c_0, with respect to the frequency, $\alpha = R(\frac{n}{\nu})^{1/2}$, is shown in figure 24. Note that the value of c_1/c_0 increases with the tube radius, R, and the square root of the frequency of oscillation, n. According to figure 24, for values of $\alpha \approx 3$, which represents a vessel of the size of the femoral artery, the magnitude of the wave velocity, $|c| \approx 0.9\ c_0$. In vessels of larger radius or at higher frequencies, $\alpha > 3$, the magnitude of the wave velocity, c, gradually increases to a value of about $0.95\ c_0$. Thus, in the larger vessels, the slowing effect of the pulse wave due to viscosity is relatively small.

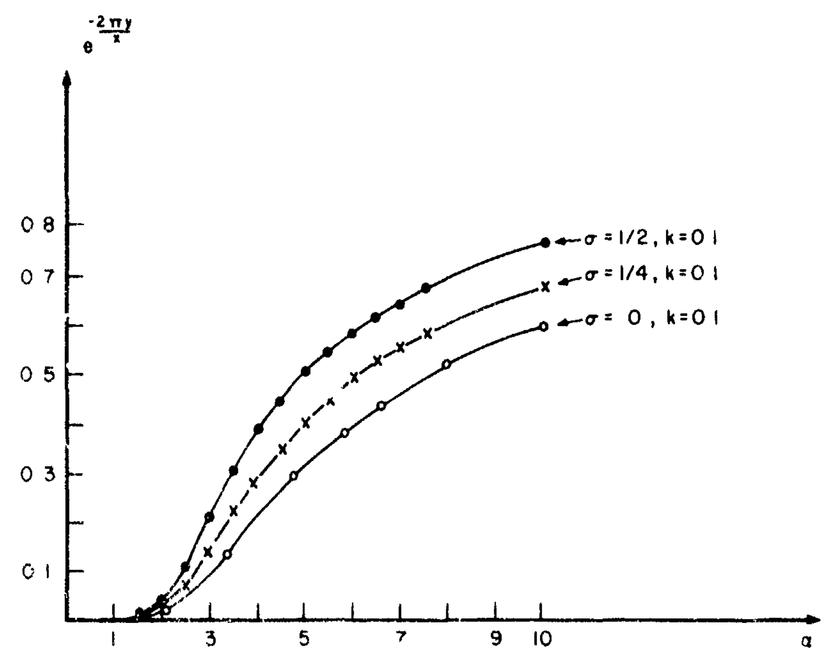

Figure 23. Variation of the damping factor with respect to α. The ordinate represents the fraction of the wave remaining after traveling over one wave-length. Note that the damping is much greater for small values of α and is 100 per cent for $\alpha < 1$.

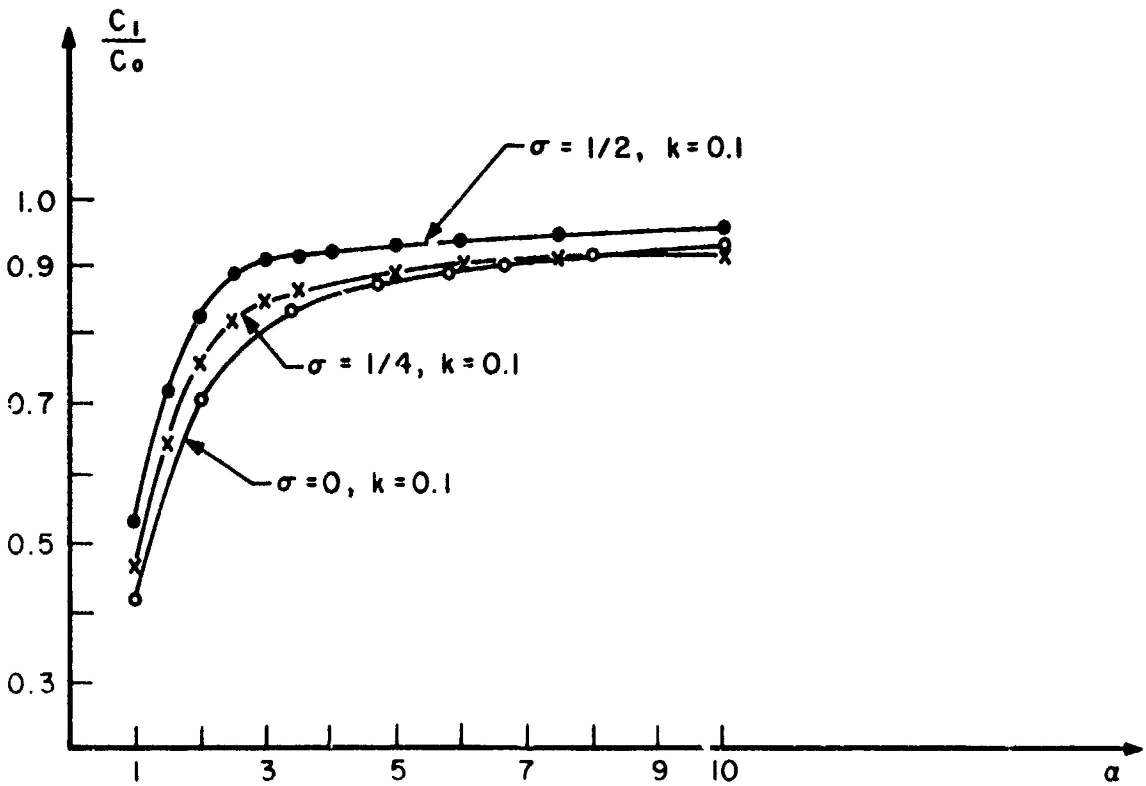

Figure 24. Variation of c_1/c_0 with respect to α. Note that for values of α greater than 3, which are those of greatest practical interest, the change in the velocity ratio, c_1/c_0, with respect to α is quite small.

From the preceding discussion, we observe that the relation of damping of the pulse wave, c, to the radius of the tube and the frequency of the wave is very important. The importance of the dependence of damping on the frequency may be seen as follows. Consider a tube of constant radius. According to figure 24, we find that in a viscous fluid the wave velocity, c, increases with α, i.e., with the frequency of oscillation, n. From figure 23 we note that the damping of the wave per wavelength decreases with α. However, with increasing frequency, the wavelength becomes shorter. Now, since

wave velocity = (wavelength)(frequency)

and the wave velocity changes with frequency, the net effect is that the wavelength always decreases with frequency.

From the practical point of view, we need also to consider damping in terms of distance, i.e., over the length of a tube. In table 1, values of the velocity ratio, c_1/c_0, are indicated for a tube length of 10 cm for the

first four harmonics of the pulse of the dog, in the femoral artery. These values are obtained for $k = h/R = 0.1$, $\sigma = 1/2$ and $\sigma = 0$. In the table, the reduction in amplitude is denoted by $f_1 = \exp\left(\frac{-2\pi Y}{X}\right)$ and $f_2 = \exp\left(\frac{-2\pi Yz}{X\lambda}\right)$ with $z = 10$ cm. Note that the percentage damping increases with the frequency even though the wave velocity increases. Thus for $\alpha = 3.34$, $\sigma = 1/2$, and the femoral artery considered as a free elastic tube, the wave velocity ratio $|c/c_0| = 0.914$ and the amplitude of the wave is damped to 27.4% of its initial value in one wavelength. This represents a damping of 5.4% in a 10-cm length. For $\alpha = 6.67$, $|c/c_0| = 0.942$, the amplitude of the wave is reduced to 63.6% in one wavelength and the damping is increased to 7.5%. One would expect that such diminuations in amplitude would have been observed and remarked upon, but until more accurate observations are available it is not possible to say with certainty that this degree of damping is greater than that which exists in the arterial system. In practical observations it might well be masked by the change in shape of the pulse as it travels.

TABLE I

The values of c_1/c_0 and damping ratios for $k = 0.1$ and a tube length of 10 cm for the first four harmonics of the pulse of the dog in the femoral artery.

	$\sigma = \frac{1}{2}$			$\sigma = 0$		
α	c_1/c_0	f_1	f_2	c_1/c_0	f_1	f_2
3.34	0.914	0.274	0.946	0.842	0.132	0.917
4.72	0.924	0.472	0.938	0.876	0.294	0.900
5.78	0.936	0.565	0.929	0.894	0.381	0.883
6.67	0.942	0.636	0.925	0.906	0.442	0.870

It is well known that the pulse wave generated by the heart contains harmonics of several frequencies. According to the above discussion, the higher frequency waves will travel faster than the lower frequency waves. Hence, the phase relations of the harmonic components will change and alter the shape of the pulse wave by dispersion. However, at the same time, the higher frequencies will be damped out first. Thus, as the pulse wave travels toward the periphery, its high frequency components will vanish. For example, the incisura of the central aortic pulse becomes damped out rapidly.

GROUP VELOCITY OF THE PULSE WAVE

Suppose the medium through which the pulse wave travels is such that the wave velocity is a function of frequency. In such a medium the wave pulses will therefore always be deformed because their different components move with different velocities. Whenever we directly measure the velocity of such a complex wave motion, in the sense that a measurement is made of the time required for the disturbance to travel a given distance, we are essentially measuring the group velocity of the wave, i.e., the velocity of the wave profile rather than the wave velocity, c.

Earlier, we had described the motion of the fluid at any instant by $e^{in(t - z/c)}$. If, instead, the motion is described by $e^{i(nt - mz)}$, upon comparison, we note that $m = nX/c_0$. From the definition of the group velocity, $c_g = dn/dm$, we find that $\frac{1}{c_g} = \frac{d(nX/c_0)}{dn} = \frac{X}{c_0} + (\frac{n}{c_0})\frac{dX}{dn}$.

Since $n = \frac{\alpha^2 \nu}{R^2}$, $\frac{dn}{d\alpha} = \frac{2\alpha\nu}{R^2}$ and $dn = (\frac{2\alpha\nu}{R^2})d\alpha$. Combining these results, we find that

$$\frac{1}{c_g} = \frac{X}{c_0} + \left(\frac{\alpha^2 \nu/R^2}{2c_0 \alpha \nu/R^2}\right)\frac{dX}{d\alpha} = \frac{X}{c_0} + \left(\frac{\alpha}{2c_0}\right)\frac{dX}{d\alpha}.$$

Since $X = c_0/c_1$, this may be written as

$$\frac{1}{c_g} = \frac{1}{c_1}\left[1 + \left(\frac{\alpha}{2X}\right)\frac{dX}{d\alpha}\right]$$

or $$\frac{c_o}{c_g} = \frac{c_o}{c_1}\left[1 + \left(\frac{\alpha}{2X}\right)\frac{dX}{d\alpha}\right] \tag{3-96}$$

In equation 3-96, the analytical form of $dX/d\alpha$ is unsuitable for computation. However, we note that if we consider the logarithms of X and α instead (see figure 25), then

$$\frac{d(\log X)}{d(\log \alpha)} = \frac{\left(\frac{1}{X}\right)dX}{\left(\frac{1}{\alpha}\right)d\alpha} = \left(\frac{\alpha}{X}\right)\frac{dX}{d\alpha}$$

Thus it is possible to estimate the magnitude of $dX/d\alpha$.

From figure 25, note that for those values of α which apply to the femoral artery, $3 < \alpha < 7$,

$$\left.\frac{d(\log X)}{d(\log \alpha)}\right]_{3 < \alpha < 7} \cong -0.045$$

and the ratio

$$\left.\frac{c_g}{c_1}\right]_{3 < \alpha < 7} \cong 0.98$$

Thus the difference between the group velocity and pulse velocity of the disturbance is approximately 2%, and over the range $\alpha = 3$ to $\alpha = 4$ is certainly never more than 2-1/2%. Until accurate measurements of pulse velocity are made over short lengths of artery, this difference is not likely to be worth taking into account.

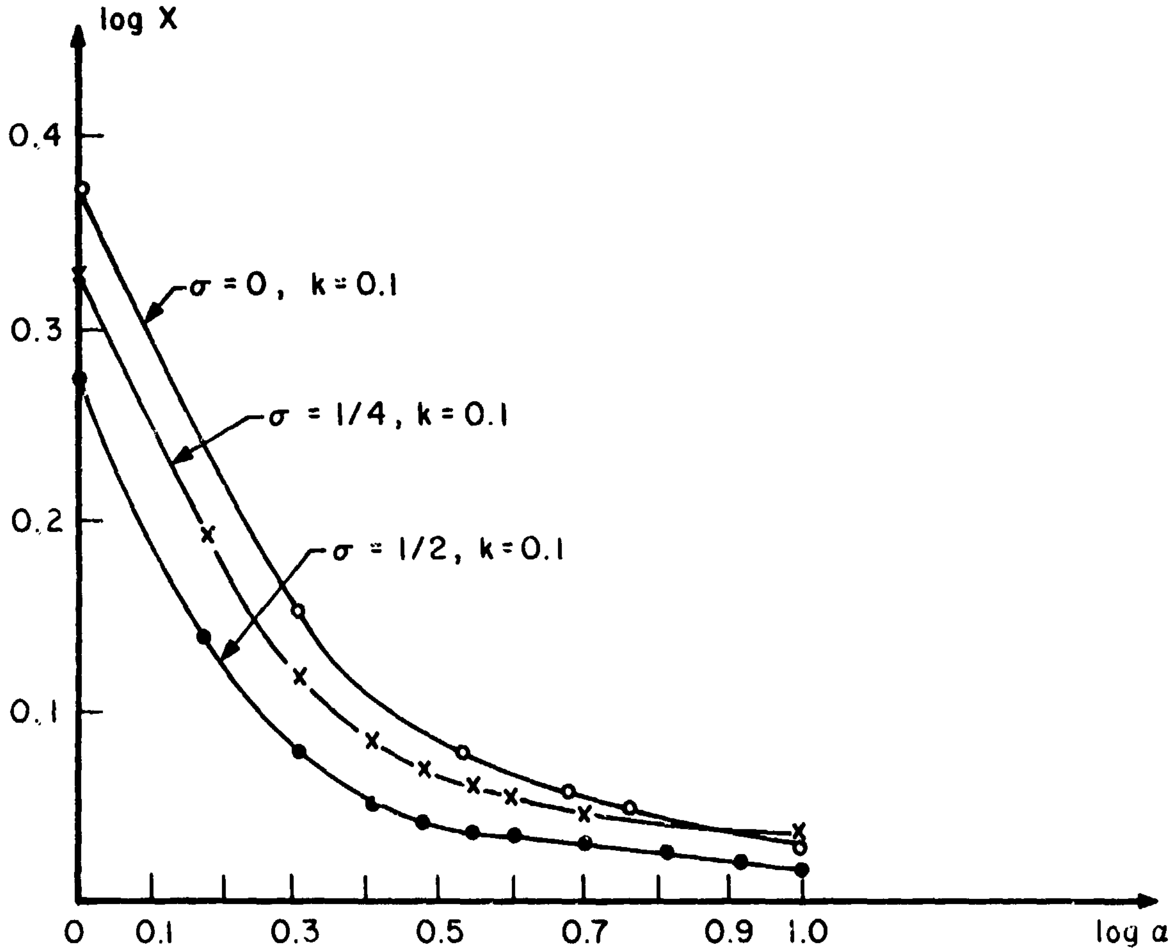

Figure 25. The variation of log X with respect to log α.

SECTION IV

EFFECTS OF MASS LOADING, TETHERING AND INTERNAL DAMPING

INTRODUCTION

The equations of the preceding chapter, describing the freely-moving elastic tube, predict longitudinal motion of the tube which is too large to be realistic. In this section we consider a more faithful representation of the mammalian arteries. To this end we modify the equations of motion of the elastic tube to take into account the additional mass of the tube which takes no part in elastic deformation, elastic constraint, since the arteries are tethered, and internal damping, since the material of the wall is not perfectly elastic. We also obtain expressions describing the phase velocity of the pressure wave and its attenuation, which includes the effect of tube wall viscosity.

TUBE WITH ADDITIONAL MASS

We have to take into account the fact that the arteries are surrounded by a tissue mass. To incorporate this reality, we assume that the additional tissue mass is uniformly distributed about the tube and takes no part in the elastic deformation. Accordingly, the inertia of the tube is increased.

In order to represent the effect of additional tissue mass, we write

$$\begin{Bmatrix}\text{effective mass of}\\ \text{artery + tissue}\end{Bmatrix} = \begin{Bmatrix}\text{mass of}\\ \text{artery}\end{Bmatrix} + \begin{Bmatrix}\text{mass of}\\ \text{tissue}\end{Bmatrix}$$

For a given length, ℓ, of tube we may write this relation as:

$$\rho_e(H'R_e\ell) = \rho(hR\ell) + \rho_1(h_1R_1\ell) \tag{4-1}$$

where
- ρ_e effective density of artery + tissue
- ρ density of artery
- ρ_1 density of tissue
- H' effective thickness of artery + tissue
- h thickness of artery
- h_1 thickness of tissue
- R_e effective radius of artery + tissue
- R_1 radius of tissue (added mass)

Assuming that

1) effective density of artery + tissue = density of artery, $\rho_e = \rho$,

2) effective radius of artery + tissue = radius of artery, $R_e = R$,

we may write equation 4-1 as

$$\rho(H'R\ell) = \rho(hR\ell) + \rho_1(h_1R_1\ell)$$

Dividing through by $\rho R\ell$, we obtain

$$H' = h\left[1 + \frac{h_1}{h}\left(\frac{\rho_1 R_1}{\rho R}\right)\right] \tag{4-2}$$

TUBE WITH ELASTIC CONSTRAINT

For a more faithful representation of the arteries <u>in situ</u>, we will now take into account the fact that the tube wall with the additional tissue mass is attached to its surroundings. We will assume that such an elastic constraint acts strictly in the longitudinal direction. The motion in the radial direction will be considered unrestricted.

We recall the equation of motion of the longitudinal displacement of the freely moving elastic tube (which includes the effects of fluid pressure and surface traction) in the form

$$\frac{\partial^2\zeta}{\partial t^2} = \left(\frac{\rho_o}{\rho}\right)\frac{\nu}{hR}\left[\frac{\partial w}{\partial y} + R\frac{\partial u}{\partial z}\right]_{y=1} + \frac{B}{\rho}\left(\frac{\partial^2\zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial\xi}{\partial z}\right) \tag{3-18}$$

If the tube is considered to be constrained along its longitudinal axis, then $\partial u/\partial z = 0$ in equation 3-18. Moreover, if we include the effect of mass-loading and longitudinal constraint, the equation of motion for the longitudinal displacement, ζ, of the tube will have the modified form

$$\frac{\partial^2\zeta}{\partial t^2} + m^2\zeta = \left(\frac{\rho_0}{\rho}\right)\frac{\nu}{hR}\left(\frac{\partial w}{\partial y}\right)_{y=1} + \frac{Bh}{H'\rho}\left(\frac{\partial^2\zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial\xi}{\partial z}\right) \tag{4-3}$$

Comparing equation 4-3 with equation 3-18, we observe the following.

1) **The second factor on the left-hand side represents the "spring effect" per unit mass due to the elastic constraint in the longitudinal direction. Note that: force due to elastic constraint per unit mass = spring effect per unit mass**

$$= \left(\frac{\text{spring constant}}{\text{mass}}\right)\left(\begin{matrix}\text{longitudinal displacement}\\ \text{of tube}\end{matrix}\right)$$

$$= \left(\begin{matrix}\text{natural circular frequency}\\ \text{of elastic constraint}\end{matrix}\right)^2\left(\begin{matrix}\text{longitudinal displacement}\\ \text{of tube}\end{matrix}\right)$$

$$= m^2\zeta.$$

2) The modification in the second term on the right-hand side of equation 4-3, as compared with the corresponding term in equation 3-18, is on account of the inclusion of the tissue mass. We recall that when the tube thickness was h, the relation between the longitudinal force, P, and the corresponding displacement which is based upon inertia concepts is of the form

$$\rho h \frac{\partial^2\zeta}{\partial t^2} = \frac{\partial P}{\partial z} \tag{3-12}$$

If the tissue mass is also taken into account, this affects the inertia of the tube and therefore equation 3-12 has to be modified to the form

$$\rho H' \frac{\partial^2\zeta}{\partial t^2} = \frac{\partial P}{\partial z} \tag{4-4}$$

where H' is the effective thickness of the tube with tissue mass. However, equation 3-10, which is not based upon inertia concepts, remains unchanged when the additional tissue mass is included. Thus, for the tube with additional tissue mass, we have

$$P = Bh\left(\frac{\partial\zeta}{\partial z} + \frac{\sigma\xi}{R}\right) \tag{3-10}$$

from which we write

$$\frac{\partial P}{\partial z} = Bh\left(\frac{\partial^2\zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial\xi}{\partial z}\right) \tag{4-5}$$

Combining equations 4-4 and 4-5, we have

$$\rho H' \frac{\partial^2 \zeta}{\partial t^2} = Bh\left(\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right)$$

or

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{Bh}{\rho H'}\left(\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R}\frac{\partial \xi}{\partial z}\right)$$

We now use equation 4-3, denoting the reduced longitudinal motion of the tube, taking into account the additional tissue mass and longitudinal constraint, instead of equation 3-18, describing the motion of the freely moving elastic tube. The equation, corresponding to equation 3-80 as a result of this replacement, is combined with equations 3-76, 3-79 and 3-81 to determine a frequency equation corresponding to equation 3-83. Performing the algebra, we find that the form of the frequency equation corresponding to equation 3-84 describing the wave velocity remains unchanged. This unchanged form is

$$(1 - \sigma^2)x^2 + 2Gx + H = 0$$

where

$$x = \frac{kB}{\rho c^2}$$

$$G = \frac{5/4 - \sigma}{1 - F_{10}} + \frac{k'}{2} + \sigma - \frac{1}{4}$$

$$H = \frac{1 + 2k'}{1 - F_{10}} - 1$$

$$k' = \left[1 + \frac{h_1}{h}\left(\frac{\rho_1 R_1}{\rho R}\right)\right]\left(1 - \frac{m^2}{n^2}\right) \tag{4-6}$$

Note that there is a difference in the description of the wall-thickness ratio, k', for the tube with additional mass and longitudinal constraint as given by equation 4-6 and the definition of the wall-thickness ratio, $k = h/R$, for the freely-moving elastic tube.

We may draw the following conclusions from equation 4-6:

1) If the frequency of oscillation of the flowing fluid is the same as the natural frequency of the tube, $n = m$, then $k' = H'/R = 0$. This implies that the thickness of the tube is zero, i.e., the mass of the tube is zero. The condition $n = m$ describes the condition of resonance.

2) If the longitudinal constraint is considered to be fairly stiff, i.e., the tube is considered to be partially restrained in the longitudinal direction, then $m > n$ and the value of k' will be finite and negative.

3) If the longitudinal constraint is considered to be very stiff, i.e., the tube is considered to be completely restrained in the longitudinal direction, then $m \gg n$ and $k' \to -\infty$.

In the original frequency equation

$$(1 - \sigma^2)(1 - F_{10})x^2 - x\{2 + k(1 - F_{10}) + F_{10}(\tfrac{1}{2} - 2\sigma)\} + F_{10} + 2k = 0 \tag{3-83}$$

we find upon expanding and dividing by k

$$(1 - \sigma^2)(1 - F_{10})\frac{x^2}{k} - \frac{2x}{k} - x(1 - F_{10}) - \frac{x}{k} F_{10}(\tfrac{1}{2} - 2\sigma) + \frac{F_{10}}{k} + 2 = 0 \tag{4-7}$$

In equation 4-7, if we read k as k' and consider the limiting condition of very stiff constraint described by $k' \to -\infty$, we find that

$$-x(1 - F_{10}) + 2 = 0$$

or

$$\frac{x}{2} = \frac{1}{1 - F_{10}} \tag{4-8}$$

Combining equation 4-8 with equation 3-94

$$\frac{c_0}{c} = [(1 - \sigma^2)\frac{x}{2}]^{1/2} \tag{3-94}$$

we find that

$$(\frac{c_0}{c})^2 = \frac{1 - \sigma^2}{1 - F_{10}} = (1 - \sigma^2)\frac{x}{2}$$

or

$$\frac{c_0}{c} = (\frac{1 - \sigma^2}{1 - F_{10}})^{1/2} \tag{4-9}$$

From the earlier relation

$$\frac{1}{1-F_{10}(\alpha)} = \left(\frac{1}{M'_{10}(\alpha)}\right) e^{-i\,\varepsilon'_{10}(\alpha)} \tag{3-88}$$

we note that

$$\left(\frac{1}{1-F_{10}(\alpha)}\right)^{1/2} = \left(\frac{1}{M'_{10}(\alpha)}\right)^{1/2} e^{-\frac{i\,\varepsilon'_{10}(\alpha)}{2}}$$

$$= \left(\frac{1}{M'_{10}(\alpha)}\right)^{1/2} \left[\cos\frac{\varepsilon'_{10}(\alpha)}{2} - i\sin\frac{\varepsilon'_{10}(\alpha)}{2}\right]$$

Therefore

$$\frac{c_o}{c} = X - iY = \left[\frac{1-\sigma^2}{1-F_{10}(\alpha)}\right]^{1/2}$$

$$= \left[\frac{1-\sigma^2}{M'_{10}(\alpha)}\right]^{1/2} \left[\cos\frac{\varepsilon'_{10}(\alpha)}{2} - i\sin\frac{\varepsilon'_{10}(\alpha)}{2}\right] \tag{4-10}$$

From equation 4-10, we find that

$$X = \left(\frac{1-\sigma^2}{M'_{10}(\alpha)} \right)^{1/2} \cos \frac{\varepsilon'_{10}(\alpha)}{2}$$

$$Y = \left(\frac{1-\sigma^2}{M'_{10}(\alpha)} \right)^{1/2} \sin \frac{\varepsilon'_{10}(\alpha)}{2}$$

If the amplitude of the wave is reduced in the ratio $\exp\{-\frac{2\pi Y}{X}\}$ for each wavelength of travel, the damping coefficient, $2\pi Y/X$, is given by

$$\frac{2\pi Y}{X} = 2\pi \tan \frac{\varepsilon'_{10}(\alpha)}{2}$$

If we take $\sigma = 1/2$, then $(1 - \sigma^2)^{1/2} = \frac{\sqrt{3}}{2}$

and $$X = \frac{\sqrt{3}}{2} \left(\frac{1}{M'_{10}(\alpha)} \right)^{1/2} \cos \frac{\varepsilon'_{10}(\alpha)}{2}$$

Moreover, the phase velocity, c_1, is given by

$$c_1 = c_0/X$$

or $$\frac{c_1}{c_0} = \frac{1}{X} = \frac{2}{\sqrt{3}} \left(M'_{10}(\alpha) \right)^{1/2} \sec \frac{\varepsilon'_{10}(\alpha)}{2} \qquad (4\text{-}11)$$

The variation of the ratio c_1/c_0 with frequency $\alpha = \left(\frac{R^2 n}{\nu}\right)^{1/2}$ is shown in figure 26 for the values $\sigma = 1/2$, $k' = 0, -2, -\infty$. For the same tube and fluid, figure 27 indicates the variation of the transmission or damping per wavelength of the pressure wave with respect to frequency, α, as it travels through the tube. Note that the value of the ratio c_1/c_0 for the constrained tube does not tend to 1 as $\alpha \to \infty$, i.e., the value of c_1 does not tend to c_0 as $\alpha \to \infty$. In fact, from equation 4-11, for $\sigma = 1/2$ and with reference to the asymptotic expansions of $M'_{10}(\alpha)$ and $\varepsilon'_{10}(\alpha)$,

$$\lim_{\alpha \to \infty} \frac{c_1}{c_0} = \lim_{\alpha \to \infty} \frac{2}{\sqrt{3}} [M_{10}(\alpha)]^{1/2} \sec \frac{\varepsilon_{10}(\alpha)}{2}$$

$$= \frac{2}{\sqrt{3}} \lim_{\alpha \to \infty} [M_{10}(\alpha)]^{1/2} \lim_{\alpha \to \infty} \sec \frac{\varepsilon_{10}(\alpha)}{2}$$

$$\cong \frac{2}{\sqrt{3}} (1)(1)$$

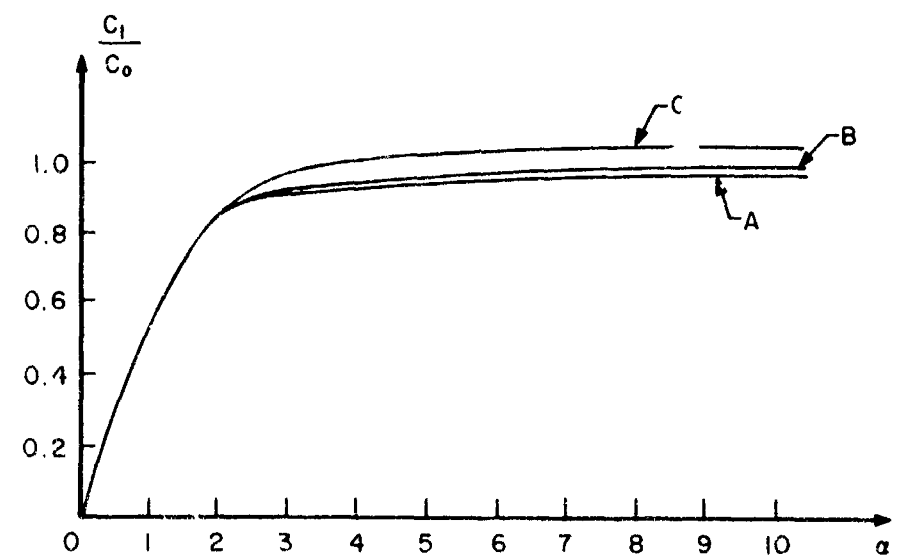

Figure 26. Variation of the phase velocity with respect to α under the following conditions of longitudinal constraint and Poisson's ratio: Curve A: $k = 0$, unconstrained tube and $\sigma = 1/2$
Curve B: $k = -2$, tube with small constraint and $\sigma = 1/2$
Curve C: $k = -\infty$, completely constrained tube and $\sigma = 1/2$

NOTE: Although the asymptotic value of c_1 for the constrained tube is $1.155\ c_0$, this value is attained very slowly, and for moderate values of α, $c_1 = c_0$ approximately for all $\alpha > 4$.

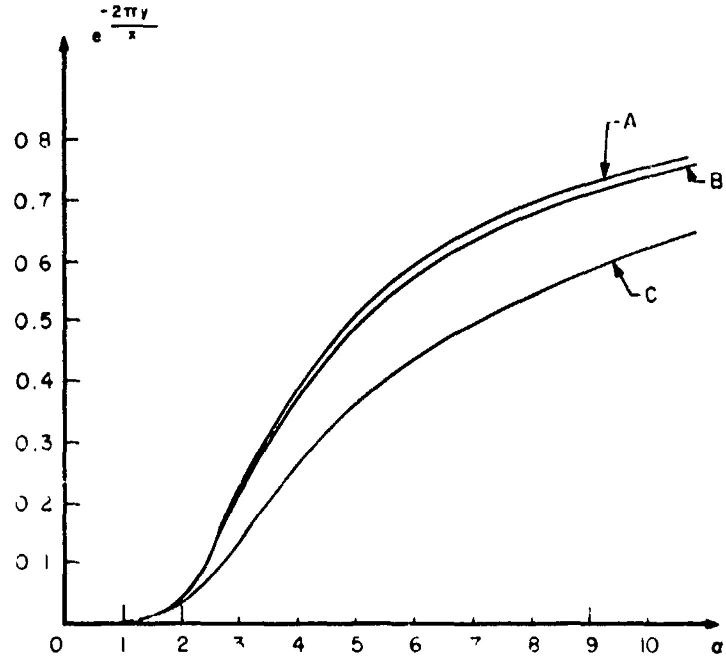

Figure 27. Variation in damping of the wave velocity with respect to α under the following conditions of longitudinal constraint and Poisson's ratio:

Curve A: $k = 0$, unconstrained tube and $\sigma = 1/2$
Curve B: $k = -2$, tube with small constraint and $\sigma = 1/2$
Curve C: $k = -\infty$, completely constrained tube and $\sigma = 1/2$

In figure 26, all three curves coincide for $\alpha < 2$ and the curve for the completely constrained tube is higher than the other two curves for $\alpha > 2$. For small values of k, the variation of c_1/c_0 with α is not sensitive to variations in k. Moreover, from figure 26, the value of the phase velocity, c_1, in a free elastic tube approaches the value c_0 as $\alpha \to \infty$. This is due to the fact that for high values of α, the motion of the fluid is determined entirely by the inertial properties, since the effects of viscosity may be neglected. We thus have a situation which corresponds to the Moens-Korteweg formula which describes the velocity, c_0, of wave transmission in an incompressible, nonviscous fluid, enclosed in a thin-walled elastic tube. According to this formula, waves of all frequencies are propagated at a constant velocity, c_0, and are not attenuated in travel along the tube.

TUBE WITH INTERNAL DAMPING

In a freely moving elastic tube, the viscous drag of the fluid would cause the tube to move in the longitudinal direction. Since this movement is not observed in the arteries, we must modify the equations of motion of the tube to account for the internal damping in the wall of the tube. To this end, we replace the elastic constants E and σ of the tube material,

which do not vary with the frequency of oscillation of the system, with complex quantities, E_c and σ_c, which vary with the frequency n.

In order to obtain the appropriate representation for E_c in terms of E, we consider the equation satisfied by longitudinal waves in an elastic bar

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$$

where $a^2 = E/\rho$. A solution of this equation is of the form

$$u(x,t) = Ae^{in\left(t - \frac{x}{a}\right)} \tag{4-12}$$

where A is an arbitrary constant. In equation 4-12, describing the longitudinal propagation of waves, if we replace E ($a^2 = E/\rho$) by E_c, where E_c is complex, the imaginary part of E_c must be positive if the motion is to be damped. Accordingly, we write the elastic quantities of the viscoelastic wall in the form

$$E_c = E[1 + in(\Delta E)] \tag{4-13}$$

$$\sigma_c = \sigma[1 + in(\Delta\sigma)] \tag{4-14}$$

From equation 4-14, $|\sigma_c| = |\sigma|\ |1 + in\,\Delta\sigma|$ and since $|1 + in\,\Delta\sigma| \geq 1$, it follows that for $\sigma = 1/2$, $|\sigma_c| \geq |\sigma| = 1/2$. As a question of principle, since Poisson's ratio for arterial tissue is known to be almost exactly 1/2, the theoretical maximum, the representation for σ_c, according to equation 4-14, may not be considered appropriate. However, if we write the representation for σ_c in the form

$$\sigma_c = \sigma e^{in\delta} \tag{4-15}$$

where δ is a parameter which measures the change in the value of σ, then

$$|\sigma_c| = |\sigma|\ |e^{in\delta}| = |\sigma|$$

In particular for $\sigma = 1/2$, $|\sigma_c| = |\sigma| = 1/2$. Of the two representations

for σ_c, equations 4-14 and 4-15, we shall use equation 4-14.

We shall now determine the effect of internal damping, i.e., the effect of changes of E and σ respectively to E_c and σ_c, on the roots of the frequency equation 3-83, i.e., on the value of c_o/c. We shall consider the following two cases: <u>Case I</u>. The effect of internal damping on the roots of the special form of the frequency equation for the limiting condition of very stiff constraint. <u>Case II</u>. The effect of internal damping on the roots of the general form of the frequency equation.

<u>Case I</u>. We know that one of the roots of the frequency equation is c_o/c. With the modifications of E and σ respectively to E_c and σ_c, it is clear that c_o is not affected, since it is a constant. However, the value of c is modified. We denote this modified value of c by c_c. As a result, we have a new ratio c_o/c_c and we may write this new ratio in terms of the old ratio c_o/c, as

$$\frac{c_o}{c_c} = \left(\frac{c_o}{c}\right)\left(\frac{c}{c_c}\right)$$

For the limiting condition of stiff constraint, we recall that

$$\left(\frac{c_o}{c}\right)^2 = (1 - \sigma^2)\,\frac{x}{2} \qquad (3\text{-}94)$$

Moreover, we recall that

$$x = \frac{hB}{R\rho_o c^2} = \left(\frac{h}{R}\right)\left(\frac{E}{1-\sigma^2}\right)\left(\frac{1}{\rho_o c^2}\right)$$

Substituting this value of x in equation 3-94 and taking $\rho = \rho_o$, we obtain

$$\frac{c_o}{c} = \left[(1-\sigma^2)\frac{hE}{2R\rho c^2(1-\sigma^2)}\right]^{1/2} = \left[\frac{hE}{2R\rho c^2}\right]^{1/2}$$

Now, the ratio c_o/c corresponds to real values of the tube parameters E and σ and the ratio c_o/c_c corresponds to the complex parameters E_c and σ_c. Moreover, from the relation

$$\frac{c_o}{c} = \left[\left(1-\sigma^2\right)\frac{x}{2}\right]^{1/2} = \frac{1}{c}\left(\frac{hE}{2R\rho}\right)^{1/2} \tag{4-16}$$

we may write for the new ratio, c_o/c_c, corresponding to the complex parameter, E_c

$$\frac{c_o}{c_c} = \frac{1}{c}\left(\frac{hE_c}{2R\rho}\right)^{1/2} \tag{4-17}$$

Taking the ratios of the corresponding sides of equations 4-16 and 4-17, we obtain

$$\frac{\dfrac{c_o}{c}}{\dfrac{c_o}{c_c}} = \frac{\dfrac{1}{c}\left(\dfrac{hE}{2R\rho}\right)^{1/2}}{\dfrac{1}{c}\left(\dfrac{hE_c}{2R\rho}\right)^{1/2}}$$

or

$$\frac{c_o}{c} = \left(\frac{c_o}{c_c}\right)\left(\frac{E}{E_c}\right)^{1/2} = \left(\frac{1-\sigma_c^2}{1-F_{10}}\right)^{1/2}\left(\frac{E}{E_c}\right)^{1/2}$$

$$= \left[\frac{1-\sigma^2(1+in\,\Delta\sigma)^2}{1-F_{10}}\right]^{1/2}\left[\frac{E}{E(1+in\,\Delta E)}\right]^{1/2}$$

$$= \left[\frac{1-\sigma^2(1+in\,\Delta\sigma)^2}{1-F_{10}}\right]^{1/2}\left[\frac{1}{1+in\Delta E}\right]^{1/2} \qquad (4\text{-}18)$$

From equation 4-18 we have, upon multiplying and dividing the right-hand side by $(1 - \sigma^2)^{1/2}$

$$\frac{c_o}{c} = \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left[\frac{1}{1-\sigma^2} - \frac{\sigma^2(1+in\,\Delta\sigma)^2}{1-\sigma^2}\right]^{1/2}\left[\frac{1}{1+in\Delta E}\right]^{1/2}$$

$$= \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left(1+in\Delta E\right)^{-1/2}\left[\frac{1}{1-\sigma^2} - \frac{\sigma^2\left\{1-n^2(\Delta\sigma)^2+2in\Delta\sigma\right\}}{1-\sigma^2}\right]^{1/2}$$

$$= \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left(1-\frac{in\,\Delta E}{2}\right)\left[\frac{1}{1-\sigma^2} - \frac{\sigma^2}{1-\sigma^2} - \frac{2in\sigma^2\Delta\sigma}{1-\sigma^2}\right]^{1/2}$$

$$= \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left(1-\frac{in\,\Delta E}{2}\right)\left[1-2in\,\Delta\sigma\left(\frac{\sigma^2}{1-\sigma^2}\right)\right]^{1/2}$$

$$= \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left(1-\frac{in\,\Delta E}{2}\right)\left[1-in\,\Delta\sigma\left(\frac{\sigma^2}{1-\sigma^2}\right)\right]$$

$$= \left(\frac{1-\sigma^2}{1-F_{10}}\right)^{1/2}\left[1-in\left(\frac{\Delta E}{2}+\frac{\sigma^2}{1-\sigma^2}\,\Delta\sigma\right)\right] \tag{4-19}$$

where we have used the binomial expansion and considered the products $n\Delta E$ and $n\Delta\sigma$ as small.

In equation 4-19, if we take $\sigma = 1/2$, then

$$(1 - \sigma^2)^{1/2} = \frac{\sqrt{3}}{2}$$

and

$$\frac{\sigma^2}{1 - \sigma^2} = \frac{1}{3}$$

Moreover, from equation 3-88, if we write

$$\left(\frac{1}{1-F_{10}}\right)^{1/2} = \left(\frac{1}{M'_{10}}\right)^{1/2} e^{-\frac{i\varepsilon'_{10}}{2}}$$

then equation 4-19 reduces to the form

$$\frac{c_c}{c} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{M'_{10}}\right)^{1/2} e^{\frac{-i\epsilon'_{10}}{2}}\left[1 - in\left(\frac{\Delta E}{2} + \frac{\Delta\sigma}{3}\right)\right] \tag{4-20}$$

Equation 4-20 describes the wave velocity, c, for complete longitudinal tethering, ($k' \to -\infty$), with complex tube parameters E_c and σ_c, $\sigma = 1/2$, and ΔE and $\Delta\sigma$ considered small.

The rest of the discussion in this section follows the work of Taylor (Taylor, 1959). For convenience, we may write equation 4-20 in the abbreviated form:

$$\frac{c_o}{c} = (X - iY)[1 - in\ N] = X - nYN - i(Y + nXN)$$

We compare this expression for c_o/c with complex tube parameters or tube with viscoelastic wall with the expression for c_o/c with elastic wall and complete longitudinal tethering

$$\frac{c_o}{c} = X - iY$$

For the elastic case, the phase velocity is given by

$$\frac{c_1}{c_0} = \frac{1}{X}$$

and for the viscoelastic case

$$\frac{c_1}{c_o} = \frac{1}{X\left(1 - \frac{nYN}{X}\right)}$$

We find that there is an increase in the phase velocity for the viscoelastic case as compared with the elastic case.

Moreover, the damping per wavelength for the elastic case is described by

$$\exp\left\{-\frac{2\pi Y}{X}\right\}$$

and for the viscoelastic case

$$\exp\left\{-\frac{2\pi Y}{X}\left[\frac{1+\frac{nXN}{Y}}{1-\frac{nYN}{X}}\right]\right\}$$

We note an increase in damping per wavelength in the viscoelastic case as compared with the elastic case.

The influence of tube-wall viscosity depends upon the value of α (i.e., the frequency) involved. See figure 28. For small values of α, the ratio Y/X is large or near unity. Generally, since the product nN is small, the difference in the phase velocities in the elastic and viscoelastic cases is small. However, for large values of α, the damping per wavelength is very sensitive to tube-wall viscosity, since, in this case, the value of the ratio Y/X decreases and hence, X/Y in the numerator of the last expression above becomes large and attenuation of the wave is increased.

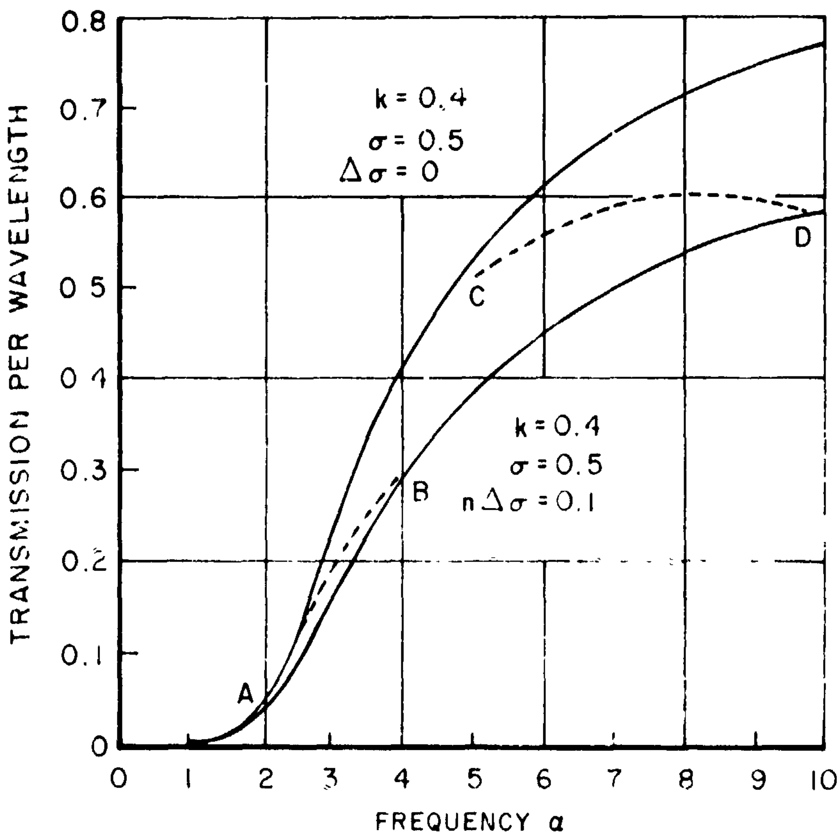

Figure 28. Transmission of the pressure wave per wavelength as a function of α in a viscoelastic tube indicating different effects of the wall-viscosity in different ranges of α. In the range $\alpha = 2$ to $\alpha = 4$, the variation of the transmission per wavelength is indicated by curve AB. In the range $\alpha = 5$ to $\alpha = 10$, the variation is described by curve CD (Taylor, 1959).

Case II. Here we start with the general form of the frequency equation (3-83) for the freely-moving elastic tube and introduce the parameters

$$E_c = E(1 + in\ \Delta E)$$

$$\sigma_c = \sigma(1 + in\ \Delta\sigma)$$

$$\sigma = 0.5,\ k = 0.4$$

In a manner similar to Case I, we find that (Taylor, 1959):

$$\left(\frac{c_0}{c}\right)^2 = \left(1-\sigma^2\right)\frac{x}{2} - in\,\Delta\sigma\left\{\frac{1}{2} + \frac{\frac{1}{3}\left(5F_{10} - 1.6\right)}{\left(1-F_{10}\right)\left[\left(1-\sigma^2\right)x - 0.45\right] - 0.75}\right\} \tag{4-21}$$

Equation 4-21, for the freely-moving tube with viscoelastic walls, replaces the equation

$$\left(\frac{c_0}{c}\right)^2 = \left(1-\sigma^2\right)\frac{x}{2} = \left(X - iY\right)^2$$

for the freely-miving elastic tube. For convenience, we consider two special cases:

(1) $\Delta\sigma = 0$

(2) $n\Delta\sigma = 0.1$

We find that (see figure 28):

A. The effect of including tube-wall viscosity on the phase velocity is negligible: For $\alpha = 1$, $\alpha = 2$, the value of c_1/c_0 is reduced by about 1%. For $\alpha > 2$, the value of c_1/c_0 is increased by less than 1%.

B. The effect of including tube-wall viscosity on damping per wavelength is considerable. In both cases ($\Delta\sigma = 0$, $n\Delta\sigma = 0.1$) the inclusion of tube-wall viscosity greatly reduces transmission, i.e., increases the attenuation of the pressure wave. We also note that the effect of tube-wall viscosity is different in different ranges of α.

Thus, we find that the inclusion of tube-wall viscosity, by restricting the longitudinal motion of the tube, has an effect similar to that of tethering.

SECTION V

MODIFIED FLUID EQUATIONS TO ACCOUNT FOR A PLASMA BOUNDARY LAYER

INTRODUCTION

In this section we shall first consider the fluid flowing in the freely moving elastic tube to be made up of two distinct layers: an inner blood layer bounded by an outer plasma layer. The velocity and velocity gradient of the fluid are obtained for both layers. At the junction of the blood and plasma layers, the viscous drags are equated to obtain matching boundary conditions. Finally, corresponding to the frequency equation of the freely moving elastic tube, we obtain a modified frequency equation to account for the plasma layer and draw some conclusions regarding the form of motion.

FLUID VELOCITY IN BLOOD AND PLASMA LAYERS

Consider the fluid flowing in the freely moving elastic tube to be made up of two distinct layers: an inner blood layer bounded by an outer plasma layer at the tube wall. More precisely, we specify that

1) for values of $y=r/R$ lying in the interval $0 \leqq y \leqq y_0$, the fluid in the tube is blood;

2) for values of $y=r/R$ lying in the interval $y_0 \leqq y \leqq 1$, the fluid in the tube is plasma. (See figure 29.)

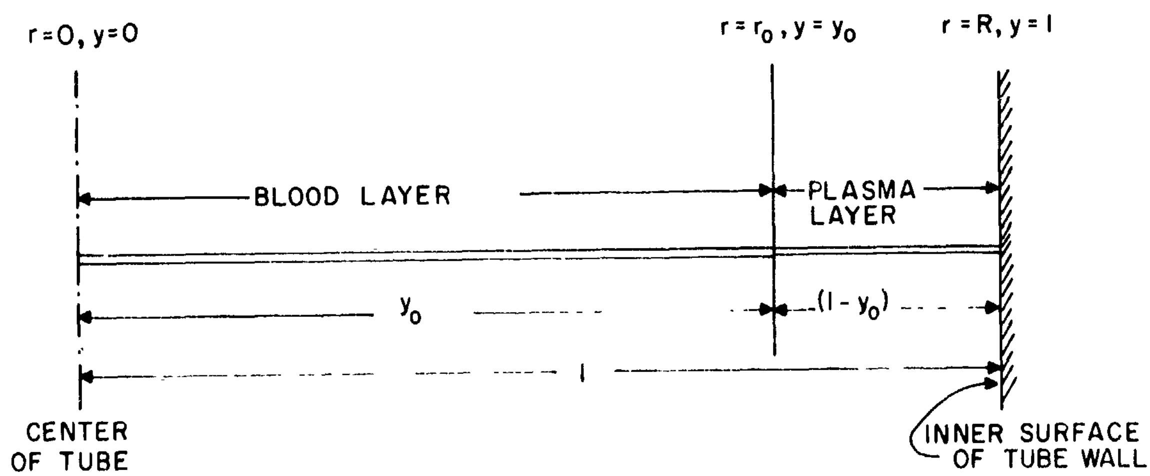

Figure 29. Blood and Plasma Layers in a Tube.

We have seen earlier (equation 3-66) that the magnitude of the longitudinal fluid velocity component is given by

$$w_1 = w_1(y) = C_1 \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left[i^{3/2}\alpha(1)\right]} + \frac{A_1}{\rho_0 c} \qquad (3\text{-}66)$$

Equation 3-66 was obtained under the condition that the fluid was blood throughout the tube, i.e., for the region $0 \leqq y \leqq 1$. Now if we restrict the region of the blood to lie within the interval $0 \leqq y \leqq y_0$, then the corresponding value of w_1 is given by

$$w_1 = w_1(y) = C_1 \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left[i^{3/2}\alpha(y_0)\right]} + \frac{A_1}{\rho_0 c} \qquad (5\text{-}1)$$

where y_0 is substituted for 1 in the denominator of the first term on the right-hand side of equation 3-66.

At the boundary between the blood and plasma layers, $y=y_0$, we shall denote the longitudinal velocity of the blood by w_0

$$w(y)\Big]_{y=y_0} = w_0$$

The value of w_0 is unknown at this time. It will be determined later from the condition that the fluid velocity must be continuous across the boundary $y=y_0$.

The magnitude of the blood velocity at the boundary is obtained by setting $y=y_0$ in equation 5-1. Thus

$$w_0 = \frac{A_1}{\rho_0 c} + C_1 \qquad (5\text{-}2)$$

Solving for the arbitrary constant, C_1, in terms of the unknown, w_0, we have from equation 5-2

$$C_1 = w_0 - \frac{A_1}{\rho_0 c}$$

Substituting this value of C_1 in equation 5-1, we obtain

$$w_1 = \frac{A_1}{\rho_0 c} + \left(w_0 - \frac{A_1}{\rho_0 c} \right) \frac{J_0\left(i^{3/2} \alpha y\right)}{J_0\left(i^{3/2} \alpha y_0\right)} \tag{5-3}$$

We shall now find a solution of the differential equation describing the magnitude of the longitudinal fluid velocity, w_1, in the plasma layer, $y_0 \leqq y \leqq 1$. We first note that the solution describing w_1 was obtained earlier in the form

$$w_1 = C_1 \frac{J_0\left(i^{3/2} \alpha y\right)}{J_0\left(i^{3/2} \alpha\right)} + \frac{A_1}{\rho_0 c} \tag{3-66}$$

This solution had the following restrictive boundary condition imposed on it: the value of w_1 stays finite at the center of the tube, y=0. This imposed condition deleted a component of the velocity from the general solution of the differential equation. This deleted component had the form

$$C_3 \frac{K_0\left(i^{1/2} \alpha y\right)}{K_0\left(i^{1/2} \alpha\right)}$$

This restrictive condition at y=0 no longer applies in the plasma layer $y_0 \leqq y \leqq 1$. In the plasma layer, the boundary conditions have to be fitted at $y=y_0$ and y=1. Accordingly, in the plasma layer, we include this deleted velocity component and write the magnitude of the longitudinal fluid velocity in the plasma layer as

$$w_1 = \frac{A_1}{\rho_0 c} + C_2 \frac{J_0(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)} + C_3 \frac{K_0(i^{1/2}\beta y)}{K_0(i^{1/2}\beta)} \tag{5-4}$$

In equation 5-4 we have introduced a nondimensional fluid parameter, β, for the plasma layer by analogy with the parameter, α, in the blood layer.

In equation 3-76, the boundary condition for w_1 had the form

$$in E_1 = C_1 + \frac{A_1}{\rho_0 c} \tag{3-76}$$

In analogy with equation 3-76, taking equation 5-4 into account, the boundary condition for the motion of the elastic tube in the plasma layer has the form

$$in E_1 = \frac{A_1}{\rho_0 c} + C_2 + C_3 \tag{5-5}$$

where C_2 and C_3 are arbitrary constants. At the boundary between the blood layer and the plasma layer, i.e., at $y=y_0$, the longitudinal fluid velocity in the plasma layer is obtained by setting $y=y_0$ in equation 5-4. Thus

$$w_1(y)\Big]_{y=y_0} = w_0 = \frac{A_1}{\rho_0 c} + C_2 \frac{J_0(i^{3/2}\beta y_0)}{J_0(i^{3/2}\beta)} + C_3 \frac{K_0(i^{1/2}\beta y_0)}{K_0(i^{1/2}\beta)} \tag{5-6}$$

VELOCITY GRADIENT IN BLOOD AND PLASMA LAYERS

The velocity gradient along the tube radius, dw_1/dy, in the blood layer is obtained by differentiating equation 5-3. Thus

$$\frac{dw_1}{dy} = w_0(-i^{3/2}\alpha)\frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} - \frac{A_1}{\rho_0 c}(-i^{3/2}\alpha)\frac{J_1(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}$$

$$= -w_0 \frac{2\left(i^{3/2}\alpha\right)}{2}\left(\frac{i^{3/2}\alpha}{i^{3/2}\alpha}\right)\left(\frac{y}{y}\right)\frac{J_1\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)}$$

$$+ \frac{A_1}{\rho_0 c}\frac{2\left(i^{3/2}\alpha\right)}{2}\left(\frac{i^{3/2}\alpha}{i^{3/2}\alpha}\right)\left(\frac{y}{y}\right)\frac{J_1\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)} \tag{5-7}$$

In analogy with the earlier notation

$$F_{10}(\alpha) = \left(\frac{2}{i^{3/2}\alpha}\right)\frac{J_1\left(i^{3/2}\alpha\right)}{J_0\left(i^{3/2}\alpha\right)}$$

we write

$$F_{10}(\alpha y) = \left(\frac{2}{i^{3/2}\alpha y}\right)\frac{J_1\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)}$$

According to this notation, we may write equation 5-7 in the form

$$\frac{dw_1}{dy} = -w_0\left(\frac{1}{2}\right)\left(i^3\alpha^2\right)y\,F_{10}(\alpha y) + \frac{A_1}{\rho_0 c}\left(\frac{1}{2}\right)\left(i^3\alpha^2\right)y\,F_{10}(\alpha y)$$

Moreover,

$$\left.\frac{dw_1}{dy}\right]_{y=y_0} = -w_0\left(\frac{i^3\alpha^2}{2}\right)y_0\,F_{10}(\alpha y_0) + \frac{A_1}{\rho_0 c}\left(\frac{i^3\alpha^2}{2}\right)y_0\,F_{10}(\alpha y_0)$$

$$= -\left(w_0 - \frac{A_1}{\rho_0 c}\right)\frac{i^3\alpha^2}{2}\, y_0\, F_{10}(\alpha y_0)$$

(5-8)

Equation 5-8 describes the velocity gradient along the tube radius in the blood layer at the boundary between the blood and plasma layers.

Differentiating the plasma fluid velocity, described by equation 5-4, we obtain the velocity gradient along the tube radius in the plasma layer. Thus

$$\left.\frac{dw_1}{dy}\right]_{y=y_0} = -C_2\left(i^{3/2}\beta\right)\frac{J_1\left(i^{3/2}\beta y_0\right)}{J_0\left(i^{3/2}\beta\right)} - C_3\left(i^{1/2}\beta\right)\frac{K_1\left(i^{1/2}\beta y_0\right)}{K_0\left(i^{1/2}\beta\right)} \qquad (5\text{-}9)$$

The velocity gradient in the plasma layer at the tube wall, y=1, is obtained by differentiating equation 5-4 and setting y=1. Thus

$$\left.\frac{dw_1}{dy}\right]_{y=1} = -C_2\left(i^{3/2}\beta\right)\frac{J_1\left(i^{3/2}\beta\right)}{J_0\left(i^{3/2}\beta\right)} - C_3\left(i^{1/2}\beta\right)\frac{K_1\left(i^{1/2}\beta\right)}{K_0\left(i^{1/2}\beta\right)} \qquad (5\text{-}10)$$

We can now determine w_0 and C_3 in terms of the arbitrary constant C_2 by

1) equating the two values of the quantity $(w_0 - A_1/\rho_0 c)$ from equations 5-4 and 5-6, i.e., by making the fluid velocity continuous at the boundary between the blood layer and the plasma layer, $y=y_0$;

2) equating the two values of the viscous drag at $y=y_0$. For convenience of writing, we introduce the following notation.

$$F_{10}(\beta) = \left(\frac{2}{i^{3/2}\beta}\right) \frac{J_1(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}$$

$$F_{10}(\beta y_0) = \left(\frac{2}{i^{3/2}\beta y_0}\right) \frac{J_1(i^{3/2}\beta y_0)}{J_0(i^{3/2}\beta y_0)}$$

$$G_{10}(\beta) = \left(\frac{2}{i^{1/2}\beta}\right) \frac{K_1(i^{1/2}\beta)}{K_0(i^{1/2}\beta)}$$

$$G_{10}(\beta y_0) = \left(\frac{2}{i^{1/2}\beta y_0}\right) \frac{K_1(i^{1/2}\beta y_0)}{K_0(i^{1/2}\beta y_0)}$$

$$F_0(\beta y_0) = \frac{J_0(i^{3/2}\beta y_0)}{J_0(i^{3/2}\beta)}$$

$$G_0(\beta y_0) = \frac{K_0(i^{1/2}\beta y_0)}{K_0(i^{1/2}\beta)}$$

With this abbreviated notation, equation 5-6 assumes the following form. From

$$w_o - \frac{A_1}{\rho_o c} = C_2 \frac{J_o(i^{3/2}\beta y_o)}{J_o(i^{3/2}\beta)} + C_3 \frac{K_o(i^{1/2}\beta y_o)}{K_o(i^{1/2}\beta)} \tag{5-6}$$

we have

$$w_o - \frac{A_1}{\rho_o c} = C_2 F_o(\beta y_o) + C_3 G_o(\beta y_o) \tag{5-11}$$

Moreover, equation 5-9 may be written as follows. From

$$\left.\frac{dw_1}{dy}\right]_{y=y_o} = -C_2(i^{3/2}\beta)\frac{J_1(i^{3/2}\beta y_o)}{J_o(i^{3/2}\beta)} - C_3(i^{1/2}\beta)\frac{K_1(i^{1/2}\beta y_o)}{K_o(i^{1/2}\beta)} \tag{5-9}$$

we have

$$\left.\frac{dw_1}{dy}\right]_{y=y_o} = -C_2\left(\frac{2i^{3/2}\beta}{2i^{3/2}\beta}\right)\left(\frac{y_o}{y_o}\right)(i^{3/2}\beta)\frac{J_o(i^{3/2}\beta y_o)}{J_o(i^{3/2}\beta y_o)}\;\frac{J_1(i^{3/2}\beta y_o)}{J_o(i^{3/2}\beta)}$$

$$- C_3\left(\frac{2i^{1/2}\beta}{2i^{1/2}\beta}\right)\left(\frac{y_o}{y_o}\right)(i^{1/2}\beta)\frac{K_o(i^{1/2}\beta y_o)}{K_o(i^{1/2}\beta y_o)}\;\frac{K_1(i^{1/2}\beta y_o)}{K_o(i^{1/2}\beta)}$$

$$= -\frac{C_2}{2} i^3\beta^2 y_o F_{10}(\beta y_o) F_o(\beta y_o) - \frac{C_3}{2} i\beta^2 y_o G_{10}(\beta y_o) G_o(\beta y_o) \tag{5-12}$$

Finally, equation 5-10 has the following form. From

$$\left.\frac{dw_1}{dy}\right]_{y=1} = -C_2\left(i^{3/2}\beta\right)\frac{J_1\left(i^{3/2}\beta\right)}{J_0\left(i^{3/2}\beta\right)} - C_3\left(i^{1/2}\beta\right)\frac{K_1\left(i^{1/2}\beta\right)}{K_0\left(i^{1/2}\beta\right)} \qquad (5\text{-}10)$$

we have

$$\left.\frac{dw_1}{dy}\right]_{y=1} = -C_2\left(\frac{2i^{3/2}\beta}{2i^{3/2}\beta}\right)\left(i^{3/2}\beta\right)\frac{J_1\left(i^{3/2}\beta\right)}{J_0\left(i^{3/2}\beta\right)} - C_3\left(\frac{2i^{1/2}\beta}{2i^{1/2}\beta}\right)\left(i^{1/2}\beta\right)\frac{K_1\left(i^{1/2}\beta\right)}{K_0\left(i^{1/2}\beta\right)}$$

$$= -C_2\left(\frac{i^3\beta^2}{2}\right)F_{10}(\beta) + C_3\left(\frac{i^3\beta^2}{2}\right)G_{10}(\beta) \qquad (5\text{-}13)$$

Equating the two values of the viscous drag at the boundary, $y=y_0$, as given by the blood and plasma layers, we have from the relation

$$\underset{\text{BLOOD}}{\mu\left.\frac{dw_1}{dy}\right]_{y=y_0}} = \underset{\text{PLASMA}}{\mu_0\left.\frac{dw_1}{dy}\right]_{y=y_0}},$$

using equations 5-8 and 5-12,

$$\mu\left[-\left(w_0 - \frac{A_1}{\rho_0 c}\right)\frac{i^3\alpha^2}{2}y_0 F_{10}(\alpha y_0)\right]$$

$$= \mu_0\left[-\frac{C_2}{2}i^3\beta^2 y_0 F_{10}(\beta y_0)F_0(\beta y_0) - \frac{C_3}{2}i\beta^2 y_0 G_{10}(\beta y_0)G_0(\beta y_0)\right]$$

$$\frac{\mu}{\mu_0}\left(w_0 - \frac{A_1}{\rho_0 c}\right) F_{10}(\alpha y_0)$$

$$= C_2\left(\frac{\beta^2}{\alpha^2}\right) F_{10}(\beta y_0) F_0(\beta y_0) + C_3\left(\frac{\beta^2}{\alpha^2}\right) G_{10}(\beta y_0) G_0(\beta y_0)$$

or

$$\frac{\mu}{\mu_0}\left(w_0 - \frac{\dot{A}_1}{\rho_0 c}\right) F_{10}(\alpha y_0)$$

$$= \frac{\beta^2}{\alpha^2}\left[C_2 F_{10}(\beta y_0) F_0(\beta y_0) + C_3 G_{10}(\beta y_0) G_0(\beta y_0)\right] \quad (5\text{-}14)$$

The parameter, α, in the blood layer is defined as

$$\alpha^2 = \frac{R^2 n}{\nu_\alpha} = \frac{R^2 n}{\mu/\rho_0} = \frac{R^2 n \rho_0}{\mu}$$

Similarly, in the plasma layer we define a parameter, β, as

$$\beta^2 = \frac{R^2 n}{\nu_\beta} = \frac{R^2 n}{\mu_0/\rho_0} = \frac{R^2 n \rho_0}{\mu_0}$$

Thus, for the ratio of these two parameters we have

$$\frac{\beta^2}{\alpha^2} = \frac{\mu}{\mu_0}$$

Using this relationship, equation 5-14 reduces to the form

$$\left(\omega_0 - \frac{A_1}{\rho_0 c}\right) F_{10}(\alpha y_0) = C_2 F_{10}(\beta y_0) F_0(\beta y_0) + C_3 G_{10}(\beta y_0) G_0(\beta y_0) \tag{5-15}$$

Combining equation 5-15 with equation 5-11,

$$F_{10}(\alpha y_0)\left[C_2 F_0(\beta y_0) + C_3 G_0(\beta y_0)\right] = C_2 F_{10}(\beta y_0) F_0(\beta y_0) + C_3 G_{10}(\beta y_0) G_0(\beta y_0)$$

or

$$C_2 F_0(\beta y_0)\left[F_{10}(\alpha y_0) - F_{10}(\beta y_0)\right] + C_3 G_0(\beta y_0)\left[F_{10}(\alpha y_0) - G_{10}(\beta y_0)\right] = 0$$

Solving for C_3

$$C_3 = -C_2 \frac{F_0(\beta y_0)}{G_0(\beta y_0)} \left[\frac{F_{10}(\alpha y_0) - F_{10}(\beta y_0)}{F_{10}(\alpha y_0) - G_{10}(\beta y_0)}\right]$$

Subsituting this value of C_3 into equation 5-5, we obtain from

$$in E_1 = \frac{A_1}{\rho_c c} + C_2 + C_3 \tag{5-5}$$

$$= \frac{A_1}{\rho_o c} + C_2 \left[1 - \frac{F_o(\beta y_o)}{G_o(\beta y_o)} \cdot \frac{F_{10}(\alpha y_o) - F_{10}(\beta y_o)}{F_{10}(\alpha y_o) - G_{10}(\beta y_o)} \right] \tag{5-16}$$

Moreover, substituting this value of C_3 into equation 5-13 we obtain from

$$\left. \frac{dw_1}{dy} \right]_{y=1} = -\frac{C_2}{2} \left(i^3 \beta^2 \right) F_{10}(\beta) + \frac{C_3}{2} \left(i^3 \beta^2 \right) G_{10}(\beta) \tag{5-13}$$

$$\left. \frac{dw_1}{dy} \right]_{y=1} = -\frac{C_2}{2} \left(i^3 \beta^2 \right) F_{10}(\beta) - \frac{C_2}{2} \left(i^3 \beta^2 \right) \frac{F_o(\beta y_o)}{G_o(\beta y_o)} G_{10}(\beta) \left[\frac{F_{10}(\alpha y_o) - F_{10}(\beta y_o)}{F_{10}(\alpha y_o) - G_{10}(\beta y_o)} \right]$$

$$= -\frac{C_2}{2} \left(i^3 \beta^2 \right) \left[F_{10}(\beta) + G_{10}(\beta) \frac{F_o(\beta y_o)}{G_o(\beta y_o)} \cdot \frac{F_{10}(\alpha y_o) - F_{10}(\beta y_o)}{F_{10}(\alpha y_o) - G_{10}(\beta y_o)} \right] \tag{5-17}$$

Note that if $y_0 \to 1$, i.e., if we consider thinner and thinner plasma layers, and, in the limit, if all the plasma is replaced by blood, then

$$\alpha = \beta, \; F_{10}(\alpha) = F_{10}(\beta)$$

and the numerator in the inner bracket in equation 5-17 is zero. Thus, equation 5-17 reduces to the form

$$\left.\frac{dw_1}{dy}\right]_{y=1} = -\frac{C_2}{2}\left(i^3\beta^2\right) F_{10}(\beta)$$

Moreover, we note that in equation

$$inE_1 = \frac{A_1}{\rho_0 c} + C_2\left[1 - \frac{F_0(\beta y_0)}{G_0(\beta y_0)} \cdot \frac{F_{10}(\alpha y_0) - F_{10}(\beta y_0)}{F_{10}(\alpha y_0) - G_{10}(\beta y_0)}\right] \quad (5\text{-}16)$$

if we let $y_0 \to 1$, then the term

$$\left.F_0(\beta y_0)\right]_{y_0=1} = \left.\frac{J_0\left(i^{3/2}\beta y_0\right)}{J_0\left(i^{3/2}\beta\right)}\right]_{y_0=1} = 1$$

Similarly,

$$\left.G_0(\beta y_0)\right]_{y_0=1} = 1$$

Thus the ratio

$$\left.\frac{F_0(\beta y_0)}{G_0(\beta y_0)}\right]_{y_0=1} = 1$$

and equation 5-16 reduces to the form

$$i n E_1 = \frac{A_1}{\rho_0 c} + C_2 \left[1 - \frac{F_{1c}(\alpha) - F_{1c}(\beta)}{F_{1c}(\alpha) - G_{10}(\beta)} \right]$$

$$= \frac{A_1}{\rho_0 c} + C_2 \left[\frac{F_{1c}(\beta) - G_{1c}(\beta)}{F_{10}(\alpha) - G_{1c}(\beta)} \right]$$

$$= \frac{A_1}{\rho_0 c} + C_2'$$

where

$$C_2' = C_2 \left[\frac{F_{1c}(\beta) - G_{10}(\beta)}{F_{1c}(\alpha) - G_{10}(\beta)} \right]$$

MODIFICATION OF THE FREQUENCY EQUATION

In the limiting condition for a vanishing plasma layer, $y_0 \to 1$, we note that the set of equations 3-76, 3-79, 3-80 and 3-81, from which we obtained the frequency equation for the freely moving elastic tube, remain the same, except that in equation 3-80 the factor $\alpha^2 F_{10}(\alpha)$ is to be replaced by the factor

$$\beta^2 F_{10}(\beta) \cdot \frac{F_{1c}(\alpha) - G_{1c}(\beta)}{F_{10}(\beta) - G_{1c}(\beta)}$$

or

$$\frac{\alpha^2 F_{1c}(\alpha)}{\alpha^2 F_{10}(\alpha)} \cdot \beta^2 F_{10}(\beta) \cdot \frac{F_{10}(\alpha) - G_{1c}(\beta)}{F_{10}(\beta) - G_{10}(\beta)}$$

or $$\alpha^2 F_{10}(\alpha)\left[\frac{\beta^2}{\alpha^2}\right] \cdot \frac{\dfrac{F_{1c}(\alpha) - G_{1c}(\beta)}{F_{1c}(\alpha)}}{\dfrac{F_{10}(\beta) - G_{10}(\beta)}{F_{10}(\beta)}}$$

or $$\alpha^2 F_{10}(\alpha)\, L(\alpha, \beta)$$

where $$L(\alpha, \beta) = \frac{\beta^2}{\alpha^2} \cdot \frac{\dfrac{F_{1c}(\alpha) - G_{10}(\beta)}{F_{1c}(\alpha)}}{\dfrac{F_{1c}(\beta) - G_{10}(\beta)}{F_{10}(\beta)}}$$

With this change in equation 3-80 we will have a corresponding change in the factors G and H in the original frequency equation (3-83). The modified frequency equation will have the form

$$(1 - \sigma^2)\, x^2 - 2G' x + H' = 0 \qquad (5\text{-}18)$$

where

$$G'(\alpha, \beta) = G' = \frac{\left(1 - \frac{\sigma}{2}\right) + L(\alpha, \beta)\left[\frac{1}{4} - \frac{\sigma}{2}\right]}{1 - F_{10}(\alpha)} + \frac{k}{2} + \frac{\sigma}{2} + L(\alpha, \beta)\left[\frac{1}{2} - \sigma\right]$$

$$H'(\alpha, \beta) = H' = L(\alpha, \beta)\left[\frac{1}{1 - F_{10}(\alpha)} - 1\right] + \frac{2k}{1 - F_{1c}(\alpha)}$$

DEDUCTIONS FROM THE MODIFIED FREQUENCY EQUATION

In the limiting conditions of heavy loading and stiff constraint, $k' \to -\infty$, the terms in $L(\alpha,\beta)$ will have no effect on the pulse velocity and damping. Although consideration of the plasma layer changes the frequency equation, for the limiting conditions of heavy loading and stiff constraint, the velocity of the pulse wave and the damping of the pulse wave in transmission are not affected.

From equation 3-86, note that

$$G\Big]_{\sigma=\frac{1}{2}} = \frac{1+\frac{1}{4}-\frac{1}{2}}{1-F_{10}(\alpha)} + \frac{k}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3/4}{1-F_{10}(\alpha)} + \frac{k}{2} + \frac{1}{4}$$

Also

$$G'\Big]_{\sigma=\frac{1}{2}} = \frac{\left(1-\frac{1}{4}\right)+L(\alpha,\beta)\left[\frac{1}{4}-\frac{1}{4}\right]}{1-F_{10}(\alpha)} + \frac{k}{2} + \frac{1}{4} + L(\alpha,\beta)\left[\frac{1}{2}-\frac{1}{2}\right]$$

$$= \frac{3/4}{1-F_{10}(\alpha)} + \frac{k}{2} + \frac{1}{4}$$

Since $G]_{\sigma=1/2} = G']_{\sigma=1/2}$, we observe that the effect of the factor $L(\alpha,\beta)$ on the roots of the frequency equation 3-82 is confined only to its effect on the factor H'.

In the modified frequency equation,

$$(1-\sigma^2)x^2 - 2G'x + H' = 0 \qquad (5\text{-}18)$$

we note that

1) the sum of the two roots = $x_1 + x_2 = \frac{2G'}{1 - \sigma^2}$

2) the product of the two roots = $x_1 x_2 = \frac{H'}{1 - \sigma^2}$

Thus the sum of the two roots will remain the same since $(1 - \sigma^2)$ and G' are the same in the original frequency equation (3-82) and the modified frequency equation (5-18). But the product of the two roots, $H'/(1 - \sigma^2)$, since it contains H', is effected by the factor $L(\alpha,\beta)$. Since

$$H' = L(\alpha,\beta)\left[\frac{1}{1 - F_{10}(\alpha)} - 1\right] + \frac{2k}{1 - F_{10}(\alpha)}$$

we find that this effect of the factor $L(\alpha,\beta)$ will be greatest when the second term, $2k/[1 - F_{10}(\alpha)]$, is zero, i.e., when $k = 0$. Thus for very thin tube walls when $k = (h/R) \to 0$,

$$H' = L(\alpha,\beta)\left[\frac{1}{1 - F_{10}(\alpha)} - 1\right]$$

The variation of the wave-velocity ratio, c_1/c_0, with α is shown in figure 30. In view of the relation

$$\frac{\mu}{\mu_0} = \frac{\beta^2}{\alpha^2}$$

we note that for $\mu = \mu_0$, i.e., for no change in the viscosity of the fluid across the cross section of the tube, we have $(\beta/\alpha) = 1$, i.e., there is no plasma boundary layer. If $(\beta/\alpha) > 1$, then we are introducing a boundary layer. Note the relationships between the wave-velocity ratio, c_1/c_0, and α as described by the two curves in figure 30, differ considerably on account of the introduction of a plasma boundary layer.

In section IV we considered a simple mathematical model based on the assumption of elastic constraint and obtained the variation of the wave-velocity ratio, c_1/c_0, with respect to α (see figure 29). Comparing

figures 29 and 30, we may conclude that if the model considered in section IV proves to be inadequate for describing the real phenomenon, then the consequences of the assumption of a boundary layer of low viscosity will have to be explored further.

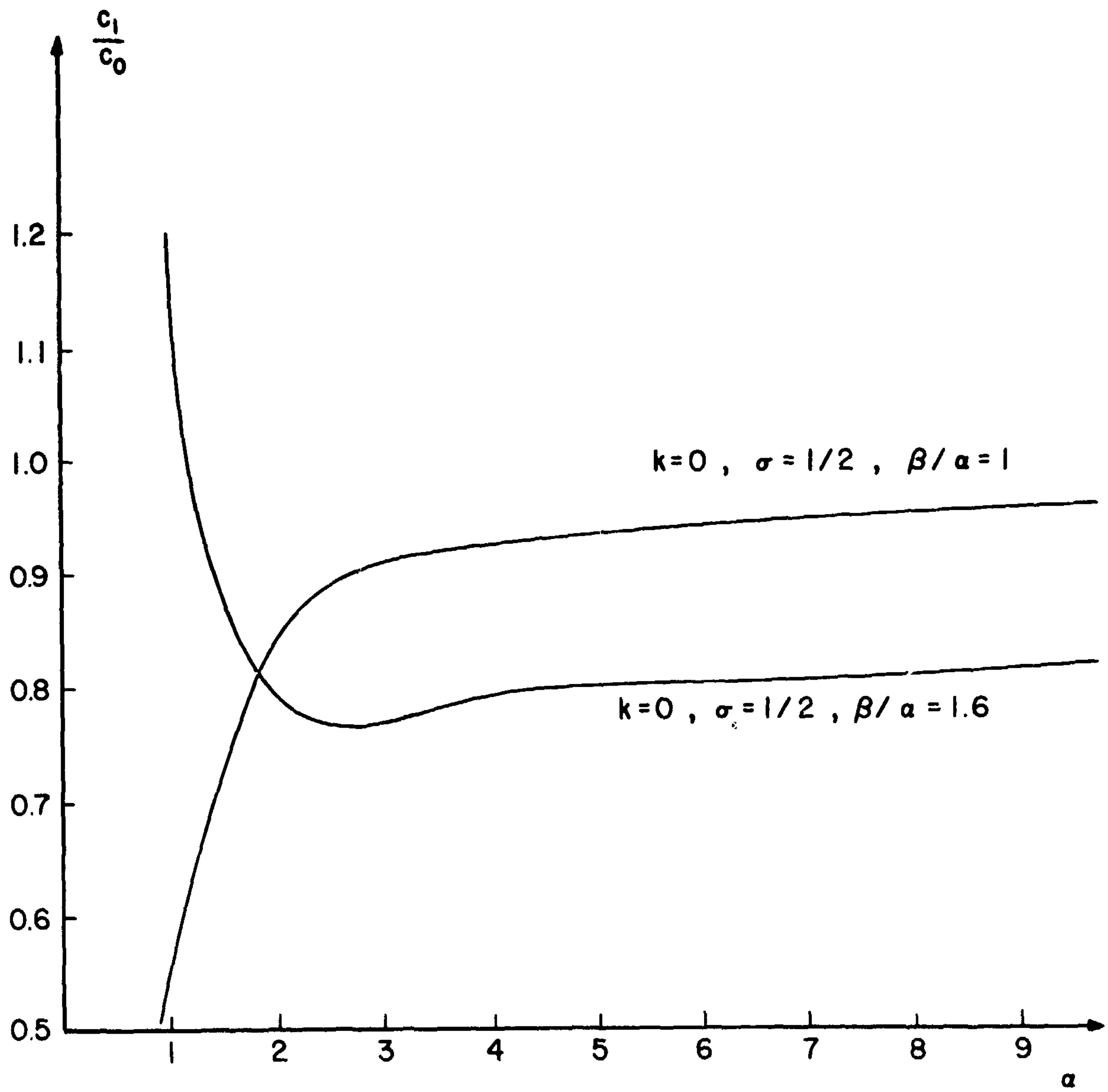

Figure 30. Variation of the velocity ratio, c_1/c_0, with α for

1) no plasma boundary layer, $\beta/\alpha = 1$;

2) a thin boundary layer described by $\beta/\alpha = 1.6$.

SECTION VI

PRESSURE-FLOW AND PRESSURE-DIAMETER RELATIONSHIPS

INTRODUCTION

In this section we shall consider the motion of the fluid over a short section of the elastic tube when subjected to a pressure gradient which is harmonic in time and the longitudinal space direction. We will first assume that there is no reflected wave present and obtain the longitudinal fluid velocity, the average fluid velocity and note the role of Poisson's ratio. Next we obtain a relation between the fluid velocity and radial expansion of the tube both in the presence and absence of a reflected wave. Finally, we note the variation of the radial expansion with internal damping of the tube wall.

MOTION OF FLUID IN ELASTIC TUBE

We shall investigate the details of the motion of the fluid over a short length of the artery over which we may consider the pressure wave velocity, c, as constant in value. We take the origin of the coordinate system at the center of this short length. (See figure 31.) We recall the assumed form of the longitudinal fluid velocity component

$$w = w_1 e^{in(t - z/c)} \tag{3-31}$$

where

$$w_1 = \frac{A_1}{\rho_0 c} + C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \tag{3-66}$$

Combining equations 3-31 and 3-66, and neglecting the value of z in equation 3-31

$$w = \left[\frac{A_1}{\rho_0 c} + C_1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right] e^{int} \tag{6-1}$$

In equation 6-1, C_1 is an arbitrary constant of integration to be evaluated from boundary conditions. A_1 is the coefficient associated with the magnitude, p_1, of the pressure, $p = p_1 \exp[in(t - z/c)]$, and having the form

$$p_1 = A_1 J_0(ky) \tag{3-48}$$

In order to relate the magnitude of the fluid pressure, A_1, to the magnitude of the longitudinal fluid velocity, w, and other known properties of the system, we substitute equations 3-76 and 3-79 into 3-81 and solve for the ratio C_1/A_1. We define this ratio according to

$$\frac{C_1}{A_1} = \frac{\eta}{\rho_0 c}$$

If we write $C_1 = (A_1/\rho_0 c)\eta$, then from equation 6-1 we obtain

$$w = \frac{A_1}{\rho_0 c}\left[1 + \eta \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right] e^{int} \tag{6-2}$$

From equation 6-2 we note that for a given applied pressure function where A_1 is known, the longitudinal fluid velocity, w, is inversely proportional to the pressure wave velocity, c.

We will now obtain the value of η in terms of σ, F_{10} and x using equations 3-76, 3-79 and 3-81. From equation

$$inE_1 = C_1 + \frac{A_1}{\rho_0 c} \tag{3-76}$$

we write
$$\frac{inE_1}{A_1/\rho_0 c} = \frac{C_1}{A_1/\rho_0 c} + 1$$

or
$$P = \eta + 1 \tag{6-3}$$

From equation

$$inD_1 = \frac{1}{2}\frac{inR}{c}\left[C_1 F_{10} + \frac{A_1}{\rho_0 c}\right] \tag{3-79}$$

we write $$\frac{inD_1}{A_1/\rho_0 c} = \frac{1}{2}\,\frac{inR}{c}\left[F_{10}\eta + 1\right]$$

or $$Q = \frac{1}{2}\,\frac{inR}{c}\left[F_{10}\eta + 1\right] \tag{6-4}$$

From equation

$$-n^2 D_1 = \frac{A_1}{\rho h} - \frac{B}{\rho}\left[\frac{\sigma}{R}\left(-\frac{inE_1}{c}\right) + \frac{D_1}{R^2}\right] \tag{3-81}$$

we write

$$-n^2 D_1 + \frac{BD_1}{\rho R^2} = \frac{A_1}{\rho h} + \frac{B\sigma}{\rho R}\left(\frac{inE_1}{c}\right)$$

or

$$\frac{inD_1}{A_1/\rho_0 c}\left[\frac{\frac{B}{\rho R^2} - n^2}{in}\right] = \frac{A_1/\rho h}{A_1/\rho_0 c} + \frac{\frac{B\sigma}{\rho Rc}\left(inE_1\right)}{A_1/\rho_0 c}$$

or

$$Q\left[\frac{\frac{B}{\rho R^2} - n^2}{in}\right] = \frac{\rho_0 c}{\rho h} + \frac{B\sigma}{\rho Rc}P \tag{6-5}$$

Substituting the values of P and Q from equations 6-3 and 6-4 into equation 6-5, we obtain

$$\frac{1}{2}\,\frac{inR}{c}\left[F_{10}\,\eta + 1\right]\left[\frac{\frac{B}{\rho R^2} - n^2}{in}\right] = \frac{\rho_0 c}{\rho h} + \frac{B\sigma}{\rho R c}\left(1+\eta\right) \qquad (6\text{-}6)$$

This is an equation in η which we have got to simplify. Multiplying equation 6-6 through by R we have

$$\frac{1}{2}\,\frac{R^2}{c}\left[F_{1c}\,\eta + 1\right]\left[\frac{B}{\rho R^2} - n^2\right] = \frac{\rho_0 c R}{\rho h} + \frac{B\sigma}{\rho c}\left(1+\eta\right) \qquad (6\text{-}7)$$

Dividing equation 6-7 by c

$$\frac{1}{2}\,\frac{R^2}{c}\left[F_{1c}\,\eta + 1\right]\left[\frac{B}{\rho R^2} - n^2\right] = \frac{\rho_0 R}{\rho h} + \frac{B\sigma}{\rho c^2}\left(1+\eta\right) \qquad (6\text{-}8)$$

Since $k = h/R$, $x = kB/\rho c^2$, the quantity n^2R^2/c^2 is considered small, and taking $\rho = \rho_0$, we may write equation 6-8 in the form

$$\frac{1}{2}\,\frac{R^2}{c^2}\left[F_{10}\,\eta + 1\right]\left[\frac{B}{\rho R^2}\right] = \frac{1}{k} + \frac{\chi\sigma}{k}\left(1+\eta\right)$$

which simplifies to

$$\frac{1-2\sigma}{2\sigma - F_{1c}} - \frac{2}{\chi\left(2\sigma - F_{1c}\right)} = \eta \qquad (6\text{-}9)$$

Now we shall identify the constant, $A_1/\rho_0 c$, appearing in equation 6-2 with the constant appearing in equation 2-19 in the simple theory of the rigid tube. We note that although the pressure wave velocity, c, in $A_1/\rho_0 c$ is a variable quantity, for a short length of the artery c is considered constant in value. We know that

$$p = p_1 e^{in(t - z/c)} = A_1 J_0(ky) e^{in(t - z/c)}$$

$$= A_1 e^{in(t - z/c)} \tag{6-10}$$

for small values of k. From equation 6-10 the pressure gradient is

$$\frac{\partial p}{\partial z} = A_1 \left(-\frac{in}{c}\right) e^{in(t - z/c)} \tag{6-11}$$

If A_1' is the coefficient associated with the pressure gradient in the simple theory of the rigid tube, then

$$-\frac{\partial p}{\partial z} = A_1' e^{int} \tag{6-12}$$

Comparing the coefficients in equations 6-11 and 6-12, we find that

$$A_1 \left(\frac{in}{c}\right) = A_1$$

and
$$\frac{A_1}{\rho_0 c} = \frac{A_1'}{in \rho_0}$$

Since $\alpha^2 = R^2 n/\nu$ where $\nu = \mu/\rho_0$, it follows that

$$\frac{A_1}{\rho_0 c} = \frac{A_1'}{in\rho_0} = \frac{A_1' R^2}{i\mu\alpha^2} \tag{6-13}$$

This is the relationship between the coefficient A_1 in the elastic tube theory and the coefficient A_1' in the rigid tube theory.

In the derivations of equations 6-14 through 6-25 below, we shall assume that there is no reflected wave, i.e., the representation for the pressure is of the form

$$p = A_1 e^{in(t - z/c)}$$

We recall that in the simple theory of the rigid tube (section II) we had

$$w_{RIGID} = \frac{A}{in\rho}\left[1 - \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)}\right]e^{int} \tag{2-19}$$

The corresponding average fluid velocity was found to be

$$\bar{w}_{RIGID} = \frac{A}{in\rho}\left[1 - \frac{2J_1\left(i^{3/2}\alpha y\right)}{i^{3/2}\alpha J_0\left(i^{3/2}\alpha\right)}\right]e^{int}$$

$$= \frac{A}{in\rho}\left[1 - F_{10}(\alpha)\right]e^{int} \tag{2-38}$$

Similarly, corresponding to the velocity

$$w_{ELASTIC} = \frac{A_1}{\rho_o c}\left[1 + \eta \frac{J_o\left(i^{3/2}\alpha y\right)}{J_o\left(i^{3/2}\alpha\right)}\right] e^{int} \tag{6-2}$$

we have the average velocity in the elastic tube

$$\bar{w}_{ELASTIC} = \frac{A_1}{\rho_o c}\left[1 + \eta\, F_{10}(\alpha)\right] e^{int} \tag{6-14}$$

From equation 6-14 we may write down the modulus and phase of $\bar{w}_{elastic}$ as

$$M''_{10}(\alpha) = \left|1 + \eta\, F_{10}(\alpha)\right|$$

$$\varepsilon''_{10}(\alpha) = \text{phase}\left\{1 + \eta\, F_{10}(\alpha)\right\}$$

Again, in analogy with the rigid tube representation

$$\bar{w}_{RIGID} = \left(\frac{MR^2}{\mu}\right)\left[\frac{M'_{10}(\alpha)}{\alpha^2}\right] \sin\left\{nt - \varphi + \varepsilon'_{10}(\alpha)\right\} \tag{2-42}$$

we may write

$$\bar{w}_{ELASTIC} = \left(\frac{MR^2}{\mu}\right)\left[\frac{M''_{10}(\alpha)}{\alpha^2}\right] \sin\left\{nt - \varphi + \varepsilon''_{10}(\alpha)\right\} \tag{6-15}$$

for a pressure gradient. $M \cos(nt - \phi)$. We can thus compare the values of $M'_{10}(\alpha)$ and $\varepsilon'_{10}(\alpha)$ for the rigid tube with $M''_{10}(\alpha)$ and $\varepsilon''_{10}(\alpha)$ for the elastic tube.

If, in the representation for the complex constant, η (equation 6-9), we substitute for x its value obtained under the condition of stiff constraint as given by equation 4-8, it follows that

$$\eta = \frac{1 - F_{10}(\alpha)}{F_{10}(\alpha) - 2\sigma} - \frac{1 - 2\sigma}{F_{10}(\alpha) - 2\sigma}$$

$$= \frac{-1\left[F_{10}(\alpha) - 2\sigma\right]}{F_{10}(\alpha) - 2\sigma}$$

$$= -1$$

Using this value, $\eta = -1$, in equation 6-14 and comparing it with equation 2-38, we conclude that the motion of the fluid in the elastic tube for the limiting condition of very stiff constraint is the same as the motion of the fluid in the rigid tube. We have thus obtained a check on the accuracy of the analysis.

The effect of the value of α on the volume rate of flow is indicated in figures 31 and 32. The variation of the ratio $\left|\frac{Q_{max}}{Q_{steady}}\right|$, i.e., the ratio of the maximum value of the oscillatory flow in either direction, $|Q_{max}|$, to that of the Poiseuille flow, $|Q_{steady}|$ for the same pressure gradient, with

respect to α is indicated in figure 31. The variation of the phase-lag with respect to α is shown in figure 32. These graphs may be compared with analogous graphs for the rigid tube in section II.

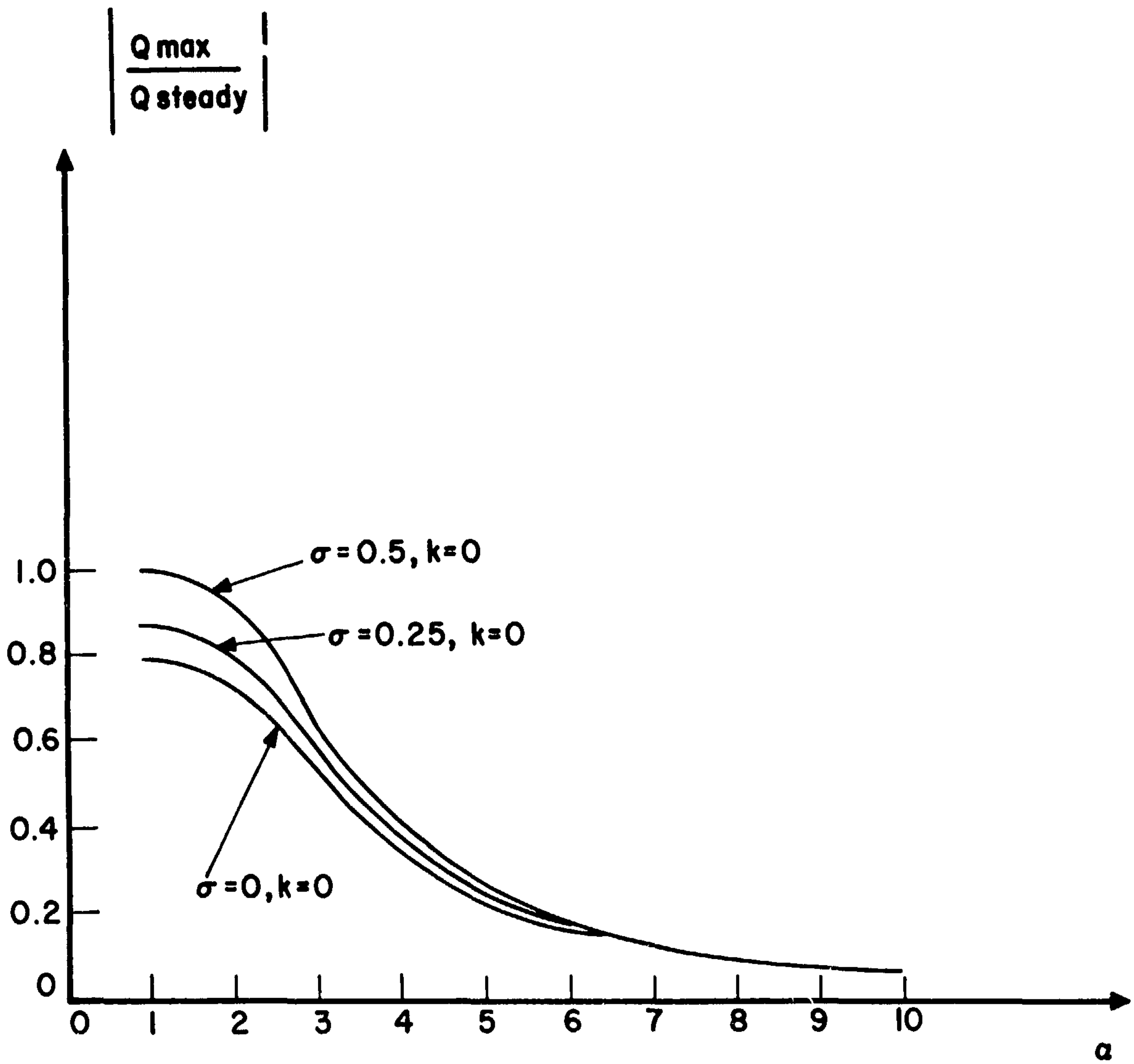

Figure 31. Variation of the amplitude ratio $\left|\frac{Q_{max}}{Q_{steady}}\right|$ with respect to α for k = 0; σ = 1/2, σ = 0. Note that for values of α < 1, the amplitude ratio approaches 1, i.e., there is little deviation from Poiseuille's formula.

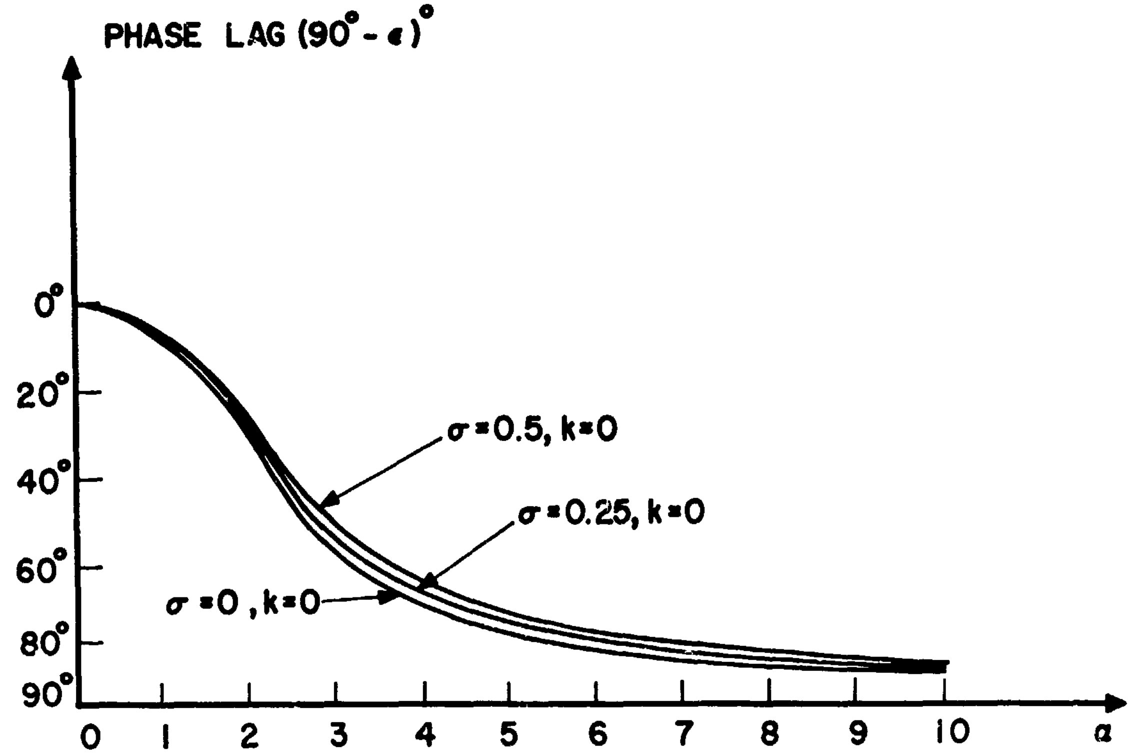

Figure 32. Variation of the phase lag, $(90° - \varepsilon)°$, of flow with respect to α for $k = 0$, $\sigma = 0$; $\sigma = 0.25$ and $\sigma = 0.5$. Note that as α increases, the phase lag approaches 90°.

The variations of the modulus of the complex fluid impedance, the fluid resistance and fluid inductance with respect to α^2 are shown in figures 33, 34 and 35 respectively. These graphs may also be compared with the corresponding graphs for the rigid tube in section II.

In order to obtain the value of the longitudinal fluid velocity in the elastic tube at the tube wall, we set $r = R$ or $y = 1$ in equation 6-2 and obtain

$$w_{ELASTIC}\Big]_{y=1} = \frac{A_1}{\rho_0 c}\left[1 + \eta\right] e^{int} \qquad (6\text{-}16)$$

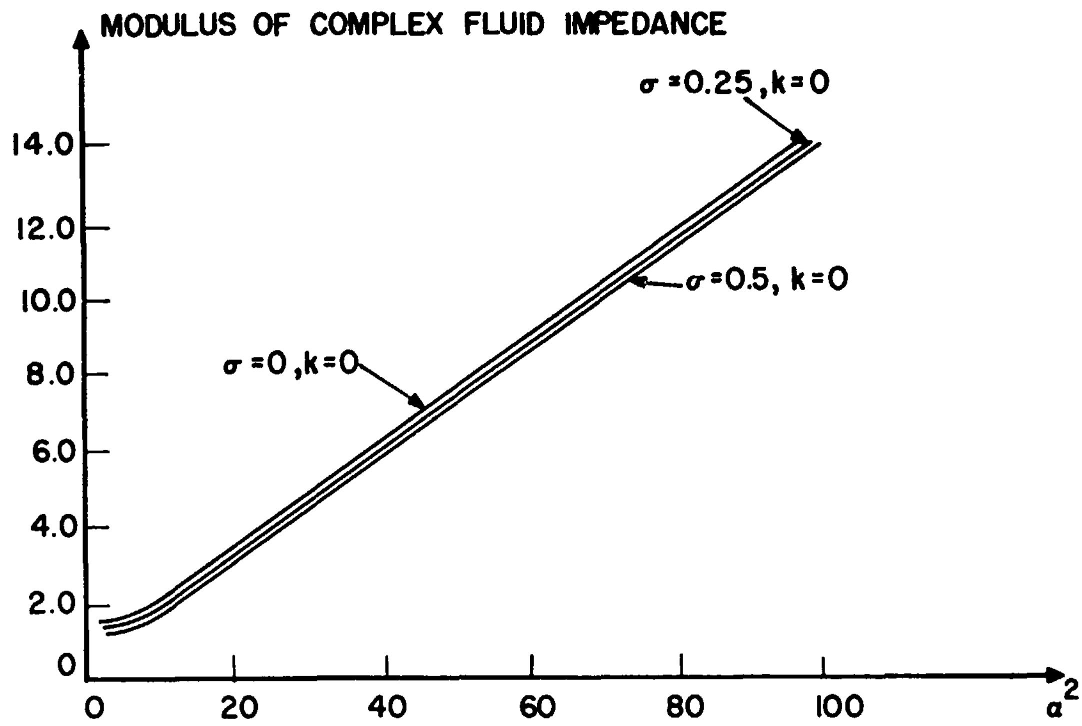

Figure 33. Variation of the modulus of the complex fluid impedance with respect to α^2 for $k = 0$; $\sigma = 0$, $\sigma = 0.25$ and $\sigma = 0.5$.

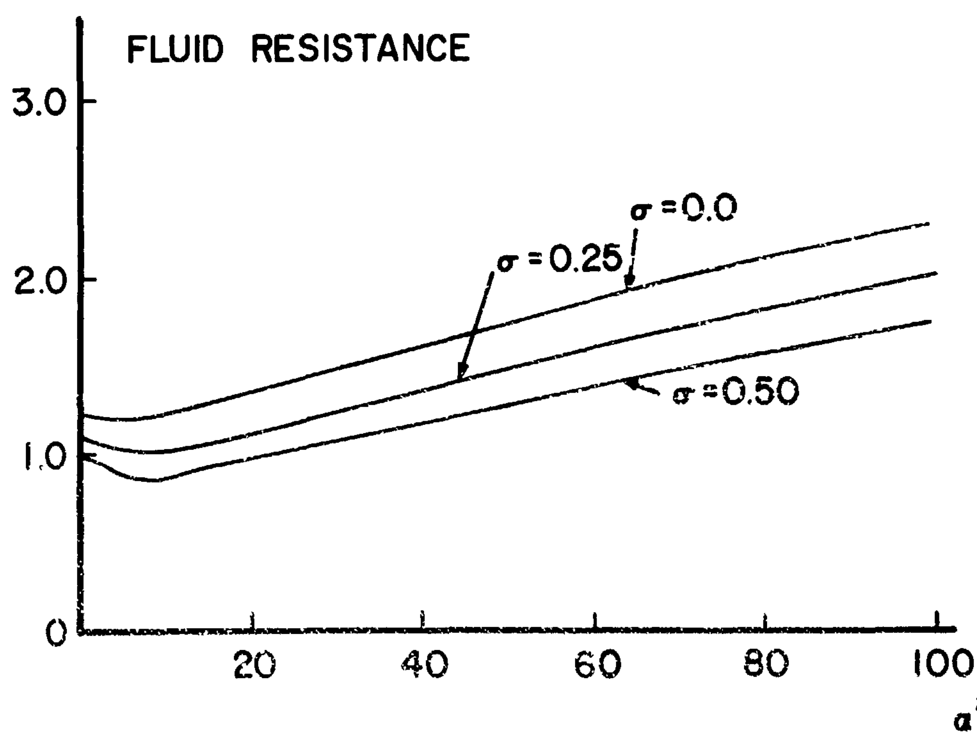

Figure 34. Variation of the fluid resistance with respect to α^2 for $k = 0$; $\sigma = 0$, $\sigma = 0.25$ and $\sigma = 0.5$.

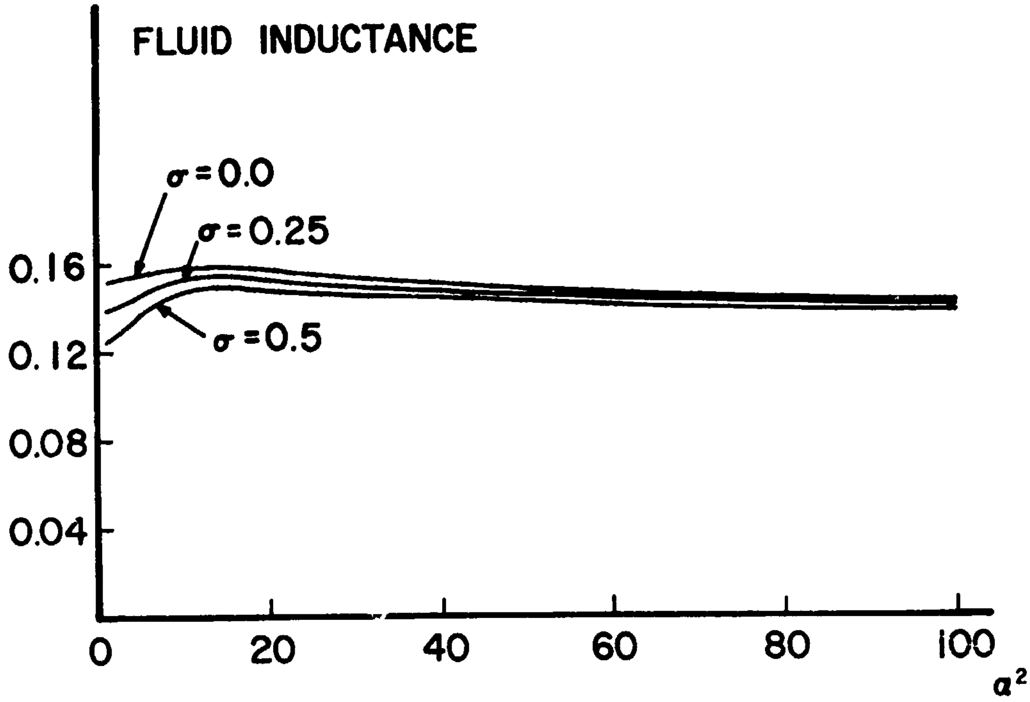

Figure 35. Variation of fluid inductance with respect to α^2 for $k = 0$; $\sigma = 0$, $\sigma = 0.25$ and $\sigma = 0.5$.

We note that if we impose the condition of very stiff constraint, $\eta = -1$, in equation 6-16, then the longitudinal fluid velocity is zero at the tube wall. From equations 6-2 and 6-16 we note that

$$\frac{\left. w_{ELASTIC} \right]_{y=1}}{\bar{w}_{ELASTIC}} = \frac{1 + \eta}{1 + \eta F_{10}(\alpha)}$$

If we calculate the modulus and phase of this ratio, they will demonstrate the ratio of the magnitude of the longitudinal fluid velocity at the tube wall to that of the average longitudinal fluid velocity in the elastic tube and the phase difference between the two. This is indicated in table II. The values of the amplitude and phase difference of the ratio $\frac{1 + \eta}{1 + \eta F_{10}(\alpha)}$ emphasize the critical role of Poisson's ratio in determining the details of

the fluid motion. The phase differences in table II are shown with their correct algebraic sign with respect to the amplitude ratio, indicating that the longitudinal fluid velocity at the wall, $w_{elastic}\big|_{y=1}$, leads the average longitudinal fluid velocity, $\bar{w}_{elastic}$. The most striking point about this effect is the magnitude of the longitudinal fluid velocity at the wall, which is greater than might be expected. It is not possible to estimate this velocity from the experimental results available. However, it would seem that the attempt to find experimental means of measuring the effect would be worth while, since it would form a critical test of the theory and would throw light on the elastic properties of the arterial wall under dynamic conditions.

TABLE II

Values of the amplitude and phase difference of the ratio $(1 + \eta)/(1 + \eta F_{10})$, comparing the velocity at the wall with the average velocity, with $F_{10} = 0.1$ and values of α corresponding to the first four harmonics of the pulse in the dog's femoral artery.

	$\sigma = 1/2$		$\sigma = 0$	
α	Amplitude Ratio	Phase Difference	Amplitude Ratio	Phase Difference
3.34	0.122	74.0°	0.527	26.2°
4.72	0.166	52.9°	0.226	16.2°
5.78	0.190	38.0°	0.145	12.6°
6.67	0.257	28.6°	0.104	10.5°

Equation 6-16 describes the longitudinal fluid velocity at the tube wall in elastic tubes. If we consider that the fluid adheres to the tube wall and integrate this equation with respect to time, we will obtain the longitudinal distance traversed by a point on the tube wall. This distance ζ is

$$\int_0^t \frac{A_1}{\rho_0 c}\left(1+\eta\right) e^{int}\, dt$$

$$= \left(\frac{A_1}{\rho c}\right)\left(\frac{1}{in}\right)\left(1+\eta\right) e^{int}$$

$$= \left(\frac{A_1'}{in\rho}\right)\left(\frac{1}{in}\right)\left(1+\eta\right) e^{int}$$

$$= -\frac{A_1'}{n^2\rho}\left(1+\eta\right) e^{int} \tag{6-17}$$

when expressed in terms of the pressure gradient. Note that we have replaced ρ_0 by ρ, since we are considering the motion of the tube wall.

Assuming $k = 0.1$, $\sigma = 0.5$, we find that the maximum value of the longitudinal displacement of the tube wall is 3.92 mm. This is greater than the diameter of the dog's femoral artery. It is reasonable to suppose that a longitudinal extension of this magnitude would have been remarked upon, had it been observed. The above calculation may seem unrealistic, since it is known that the artery is not, in practice, completely free, but it does show that the elastic constraint is not likely to be in resonance with the pulse frequency.

Equation 6-14 may be written in modulus and phase form as

$$\bar{w}_{ELASTIC} = \left[\left(\frac{A_1}{\rho_o c} \right) M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)} \right] e^{int}$$

$$= \frac{p_1}{\rho_o c} M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)}\, e^{int} \tag{6-18}$$

since $p_1 = A_1 J_0(ky) = A_1$ for $k << 1$. Since $c_0/c = X - iY$, we may write equation 6-18 in the form

$$\bar{w}_{ELASTIC} = \frac{p_1}{\rho_o c_o} \left(X - iY \right) M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)}\, e^{int} \tag{6-19}$$

Now, if we express the complex quantity $(X - iY)$ in modulus and phase form

$$X - iY = |X - iY| \text{ phase } (X - iY)$$

we note that the effect of the damping of the wave in transmission is to reduce the phase-advance of flow over pressure. We may write

$$\bar{w}_{ELASTIC} = \frac{p_1}{\rho_o c_o} |X - iY| M''_{10}(\alpha)\, e^{i\left[\varepsilon''_{10}(\alpha) + nt + \text{phase}(X - iY)\right]} \tag{6-20}$$

Note that in the case of the rigid tube, equation 2-40, the complex quantity $(X - iY)$ is not involved. Therefore, there is no change in phase for $\bar{w}_{rigid}$

Again, starting with equation 6-14,

$$\bar{w}_{\text{ELASTIC}} = \frac{A_1}{\rho_o c}\left[1 + \eta F_{10}\right] e^{int} \tag{6-14}$$

we write it in the following form for comparison with equation 6-20

$$\bar{w}_{\text{ELASTIC}} = \frac{A_1}{\rho_o c_o}\left(\frac{c_o}{c}\right)\left[1 + \eta F_{10}\right] e^{int} \tag{6-21}$$

Imposing the condition of very stiff constraint on equation 6-21, i.e., setting

$$\frac{c_o}{c} = \frac{\left(1 - \sigma^2\right)^{1/2}}{\left[1 - F_{10}\right]^{1/2}} \tag{4-9}$$

and $\eta = -1$, we obtain

$$\bar{w}_{\substack{\text{ELASTIC}\\ \text{STIFF CONSTRAINT}}} = \frac{A_1}{\rho_o c_o} \cdot \frac{\left(1-\sigma^2\right)^{1/2}}{\left(1-F_{10}\right)^{1/2}} \left(1 - F_{10}\right) e^{int}$$

$$= \frac{A_1}{\rho_o c_o}\left(1-\sigma^2\right)^{1/2}\left(1 - F_{10}\right)^{1/2} e^{int} \tag{6-22}$$

Since $$\left(1-\sigma^2\right)\Big]_{\sigma=1/2} = \frac{\sqrt{3}}{2}$$

and $$\left[1-F_{10}(\alpha)\right]^{1/2} = \left[M'_{10}(\alpha)\right]^{1/2} e^{\frac{i\varepsilon'_{10}(\alpha)}{2}}$$

we may write equation 6-22 in the form

$$\bar{\omega}_{\substack{\text{ELASTIC} \\ \text{STIFF CONSTRAINT}}} = \frac{A_1}{\rho_0 c_0}\left(\frac{\sqrt{3}}{2}\right)\left[M'_{10}(\alpha)\right]^{1/2} e^{\frac{i\varepsilon'_{10}(\alpha)}{2}} e^{int} \tag{6-23}$$

Comparing equations 6-19 and 6-23, we conclude that under the conditions of elasticity, together with stiff constraint, the maximum phase-lead of flow over pressure will be 45°. Note that the value of c_1 is not directly measurable. If we measure the pulse velocity, c_1, over a short length of the artery, we would expect to obtain a value given by

$$c_1 = \frac{c_0}{X}$$

In terms of the measured pulse velocity, c_1, equation 6-19 may be written as

$$\bar{\omega}_{\text{ELASTIC}} = \frac{p_1}{\rho_0 c_1}\left(1-\frac{iY}{X}\right) M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)} e^{int} \tag{6-24}$$

and equation 6-23 may be written as

$$\bar{\omega}_{\substack{\text{ELASTIC} \\ \text{STIFF CONSTRAINT}}} = \frac{A_1}{\rho_0 c_1 X}\left(\frac{\sqrt{3}}{2}\right)\left[M_{10}'(\alpha)\right]^{1/2} e^{\frac{i\epsilon_{10}'(\alpha)}{2}} e^{int} \qquad (6\text{-}25)$$

These equations may be experimentally verified as follows. First, we obtain a Fourier analysis of the pressure, pressure gradient and fluid velocity. Next, we abandon all preconceived ideas regarding the values of the internal radius of the artery and the viscosity of the blood, and determine the value of α that gives the best fit between pressure gradient and fluid velocity. This can be done without introducing the pulse velocity. Finally, we assume that, taking the same value of α, the fit of equation 6-23 to the observed flow curve could be tested, with the same value of c_0 for all harmonics.

Now we shall include a reflected wave in the expression for the pressure. Let the incident pressure wave be denoted by

$$p_I = A_1 \; e^{in(t - z/c_1)}$$

or

$$p_I = A_1 \; e^{i(nt - k_1 z)}$$

where $k_1 = n/c_1$.

This incident wave is incident at the point $z = 0$, where a partial reflection and transmission takes place. In other words

$$p_I = A_1 \; e^{i(nt - k_1 z)} \qquad \text{for } z < 0$$

For the reflected wave we have

$$p_R = A_2 \; e^{i(nt + k_1 z)} \qquad \text{for } z < 0$$

For the transmitted wave we have

$$p_T = A_T \; e^{i(nt - k_2 z)} \qquad \text{for } z > 0$$

where $k_2 = n/c_2$. Note that the frequency, n, remains constant. For the resultant wave motion we write

$$p = A_1 \, e^{i(nt_1 - k_1 z)} + A_2 \, e^{i(nt + k_1 z)} \qquad \text{for } z < 0 \tag{6-26}$$

$$p = A_T \, e^{i(nt - k_2 z)} \qquad \text{for } z > 0 \tag{6-27}$$

With respect to the representations 6-26 and 6-27, the following continuity conditions are imposed:

1) the pressure, p, remain continuous at the point $z = 0$;

2) the pressure gradient, $\partial p/\partial z$, remain continuous at the point $z = 0$.

The continuity of p means that the amplitudes be related as

$$A_1 + A_2 = A_T \qquad \text{at } z = 0 \tag{6-28}$$

The continuity of $\partial p/\partial z$ means that

$$k_1(A_1 - A_2) = k_2 A_T \qquad \text{at } z = 0 \tag{6-29}$$

Taking the ratio of the corresponding sides of equations 6-28 and 6-29, we have

$$\frac{A_1 - A_2}{A_1 + A_2} = \frac{k_2}{k_1}$$

or

$$\frac{c_2}{c_1} = \frac{A_1 - A_2}{A_1 + A_2}$$

Thus, due to a reflected wave, the pressure is reduced according to the ratio $(A_1 - A_2)/(A_1 + A_2)$. If there is no reflected wave, then

$$\frac{A_1 - A_2}{A_1 + A_2} = 1$$

i.e.,

$$A_2 = 0$$

According to the above discussion, we may write the earlier equation

$$\bar{w}_{\substack{\text{ELASTIC}\\ \text{STIFF CONSTRAINT}\\ \text{NO REFLECTED WAVE}}} = \frac{A_1}{\rho_0 c_0}\left(\frac{\sqrt{3}}{2}\right)\left[M'_{10}(\alpha)\right]^{1/2} e^{\frac{i\,\varepsilon'_{10}(\alpha)}{2}} e^{int} \tag{6-23}$$

in the form

$$\bar{w}_{\substack{\text{ELASTIC}\\ \text{STIFF CONSTRAINT}\\ \text{REFLECTED WAVE PRESENT}}} = \frac{A_1}{\rho_0 c_0}\left(\frac{A_1 - A_2}{A_1 + A_2}\right)\frac{\sqrt{3}}{2}\left[M_{10}'(\alpha)\right]^{1/2} e^{\frac{i\varepsilon_{10}'(\alpha)}{2}} e^{int}$$

$$= \frac{A_1}{\rho_0 c_0'}\left(\frac{\sqrt{3}}{2}\right)\left[M_{10}'(\alpha)\right]^{1/2} e^{\frac{i\varepsilon_{10}'(\alpha)}{2}} e^{int}$$

Thus, when a reflected wave is present we have to account for an "apparent" velocity, c_0', defined by

$$c_0' = c_0\left(\frac{A_1 + A_2}{A_1 - A_2}\right) \tag{6-30}$$

The amplitudes A_1 and A_2 appearing in equation 6-30 must be considered as complex, since no phase constants were included in the description of the incident wave, p_I, and the reflected wave, p_R, above.

If we assume that the theory developed is correct, then the best use we can make of an analysis of simultaneous recordings of pressure and pressure gradient would be to obtain information about any reflected wave that may be present. To test the validity of the theory itself, some means would

have to be found for suppressing the reflected wave. One method might be to apply a matching terminal impedance at the next junction on the distal side of the point of measurement.

Another experimental test of the theory can be devised which is free from this difficulty, using the relationship between pressure and radial expansion. This is considered in the next section.

RELATIONSHIP BETWEEN FLUID VELOCITY AND RADIAL EXPANSION

Now we shall obtain a relationship between the average fluid velocity and the radial expansion of the tube. First, for the simpler condition when there is no reflected wave, we note that equation 3-79 may be written as

$$in D_1 = \left(\frac{inR}{2c}\right)\left\{\frac{A_1}{\rho_o c} + C_1 F_{10}(\alpha)\right\} \tag{3-79}$$

$$= \left(\frac{inR}{2c}\right)\left\{\frac{A_1}{\rho_o c} + \frac{A_1}{\rho_o c}\,\frac{C_1}{A_1/\rho_o c}\,F_{10}(\alpha)\right\}$$

$$= \left(\frac{inR}{2c}\right)\frac{A_1}{\rho_o c}\left\{1 + \eta\, F_{10}(\alpha)\right\}$$

$$= \left(\frac{inR}{2c}\right)\bar{w}_{1\,\text{ELASTIC}} \tag{6-31}$$

using equation 6-14. Since the radial expansion, ξ, is given by

$$\xi = D_1\, e^{in(t - z/c)} \tag{3-72}$$

we note that for small values of z we have

$$\xi = D_1\, e^{int} \tag{6-32}$$

Combining equations 6-31 and 6-32, we find that

$$\frac{in\xi}{e^{int}} = \left(\frac{inR}{2c}\right)\bar{w}_1$$

or

$$\frac{2\xi}{R} = \frac{\bar{w}_1}{c}\, e^{int}$$

or

$$\frac{2\xi}{R} = \frac{\bar{w}}{c} \tag{6-33}$$

We will now show that equation 6-33 can be obtained directly from the continuity equation (3-24)

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{3-24}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = -\frac{\partial w}{\partial z} \tag{6-34}$$

Setting $y = r/R$, we find from equation 6-34 that

$$\left(\frac{1}{Ry}\right)\frac{1}{R}\frac{\partial}{\partial y}(Ryu) = -\frac{\partial w}{\partial z}$$

or

$$\frac{u}{Ry} = -\frac{\partial w}{\partial z}$$

or

$$u = -Ry\frac{\partial w}{\partial z} \tag{6-35}$$

Integrating equation 6-35 with respect to y from y=0 to y=1,

$$\int_{y=0}^{y=1} u\, dy = -R\int_{y=0}^{y=1}\left(\frac{\partial w}{\partial z}\right) y\, dy$$

$$uy\Big]_{y=0}^{y=1} = -R\,\frac{\partial w}{\partial z}\,\frac{y^2}{2}\Big]_{y=0}^{y=1}$$

$$u = -\frac{R}{2}\frac{\partial w}{\partial z} \qquad \text{FOR } y = 1,$$

i.e., radial fluid velocity at tube wall $= -\frac{R}{2}\left(\frac{\partial \bar{w}}{\partial z}\right)$. For no slip at the tube wall we may write the last statement as

$$u = \frac{\partial \xi}{\partial t} = -\frac{R}{2}\left(\frac{\partial \bar{w}}{\partial z}\right)$$

or

$$\frac{\partial \bar{w}}{\partial z} = -\frac{2}{R}\frac{\partial \xi}{\partial t} \qquad (6\text{-}36)$$

Moreover, since

$$\frac{\partial \bar{w}}{\partial z} = -\frac{1}{c}\frac{\partial \bar{w}}{\partial t} \qquad (6\text{-}37)$$

by combining equations 6-36 and 6-37, we have

$$-\frac{2}{R}\frac{\partial \xi}{\partial t} = -\frac{1}{c}\frac{\partial \bar{w}}{\partial t} \qquad (6\text{-}38)$$

Integrating equation 6-38 with respect to time, we obtain

$$\frac{2\xi}{R} = \frac{\bar{w}}{c} \qquad (6\text{-}39)$$

Now we will consider the case when the expression describing the pressure contains a reflected wave, i.e., the pressure has the representation

$$p = A_1\, e^{in(t - z/c)} + A_2\, e^{in(t + z/c)} \qquad (6\text{-}40)$$

Then the corresponding average fluid velocity when a reflected wave is present is, in analogy with equation 6-19, given by

$$\bar{w}_{ELASTIC} = \frac{1}{\rho_0 c}\left[A_1 e^{in(t - z/c)} + A_2 e^{in(t + z/c)}\right] M''_{10}(\alpha)\, e^{i\epsilon''_{10}(\alpha)} \qquad (6\text{-}41)$$

Combining equations 6-39 and 6-41, we write

$$\frac{2\xi}{R} = \frac{p}{\rho_0 c^2} M''_{10}(\alpha)\, e^{i\epsilon''_{10}(\alpha)} \qquad (6\text{-}42)$$

where p is given by equation 6-40. We may write equation 6-42 in the form

$$\frac{2\xi}{R} = \frac{p}{\rho_o c_o^2}\left(\frac{c_o^2}{c^2}\right)\left[M_{10}''(\alpha)\, e^{i\epsilon_{10}''(\alpha)}\right] \tag{6-43}$$

Since M_{10}'' and ϵ_{10}'' are the modulus and phase of the quantity $(1 + \eta F_{10})$, and

$$\frac{c_0^2}{c^2} = (1 - \sigma^2)\,\frac{x}{2} \tag{3-90}$$

we may write equation 6-43 as

$$\frac{2\xi}{R} = \frac{p}{\rho_o c_o^2}\left[(1-\sigma^2)\,\frac{x}{2}\right]\left[1 + \eta F_{10}\right] \tag{6-44}$$

The form of equation 6-44 is the same whether a reflected wave is present or not.

In equation 6-44, we note that the limiting value of the quantity $\frac{x}{2}(1 + \eta F_{10})$ for the case of very stiff constraint is obtained by setting $\eta = -1$ and $x = \frac{2}{1 - F_{10}}$. Thus

$$\frac{x}{2}\left(1 + \eta F_{10}\right) = \left(\frac{2}{1-F_{10}}\right)\left(\frac{1}{2}\right)\left(1 - F_{10}\right) = 1$$

Therefore, for the limiting condition of very stiff constraint, the quantity $\frac{x}{2}(1 + \eta F_{10})$ has actually the real value 1 and its phase is zero. See figure 36. However, we add a small positive imaginary component to account for damping due to the viscosity of the fluid. From equation 6-44 we note that

$$\left(\frac{2}{R}\right)\text{phase}\left\{\xi\right\} = \left(\frac{1-\sigma^2}{\rho_o c_o^2}\right)\left[\text{phase}\left\{p\right\} + \text{phase}\left\{\frac{x}{2}\left(1 + \eta F_{10}\right)\right\}\right] \tag{6-45}$$

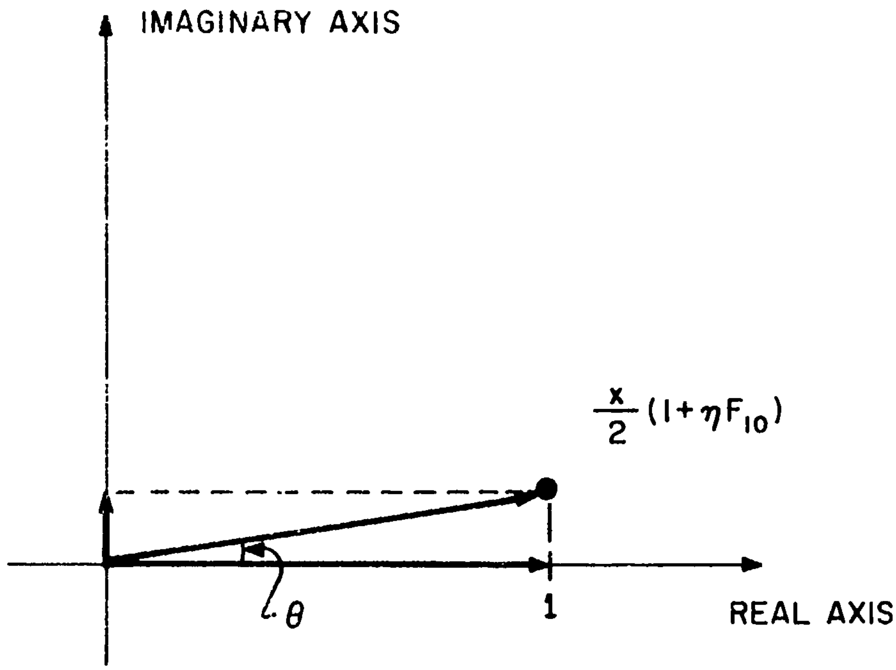

Figure 36. Small positive phase of the complex quantity $\frac{x}{2}$ $(1 + \eta F_{10})$

For all finite values of k, the phase of the quantity $\frac{x}{2}$ $(1 + \eta F_{10})$ is positive. Therefore, from equation 6-45 we conclude that the phase of the variation in diameter, ξ, always leads the phase of the pressure, p, by a few degrees. See figure 37.

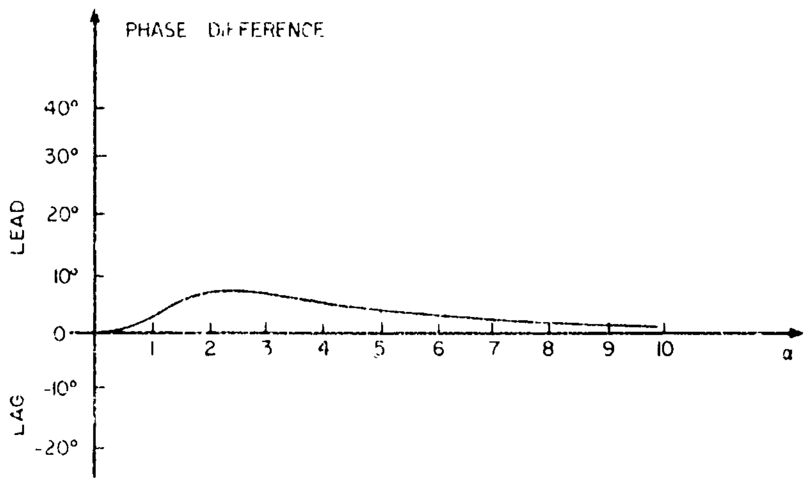

Figure 37. Variation of the phase difference (between the cyclic variations in pressure and diameter) with respect to α, for k=0 and =0.5.

RELATIONSHIP BETWEEN RADIAL EXPANSION AND INTERNAL DAMPING

Equation 6-44, in the limiting condition of heavy loading and stiff constraint, i.e.,

$$\frac{x}{2}(1 + \eta F_{10}) \rightarrow 1$$

reduces to the form

$$\frac{2\xi}{R} = \frac{p}{\rho_0 c_0^2}(1 - \sigma^2) \tag{6-46}$$

For $\sigma = 1/2$, equation 6-46 reduces to the form

$$\frac{2\xi}{R} = \frac{3}{4}\frac{p}{\rho_0 c_0^2} \tag{6-47}$$

From equation 6-46 we note that

$$\left(\frac{2}{R}\right) \text{phase}\{\xi\} = \left[\frac{1 - \sigma^2}{\rho_0 c_0^2}\right] \text{phase}\{p\}$$

Thus the pressure, p, and the radial deformation, ξ, will be in phase at all frequencies.

Since, for other conditions of constraint, the phase difference between pressure, p, and radial deformation, ξ, is always small, equation 6-46 does not provide a critical means of distinguishing between them. However, equation 6-46 does provide a test for the presence of internal damping in the tube wall. This is shown as follows.

In analogy with equation 4-19, for the elastic tube we write

$$\frac{c_c^2}{c^2} = \left(\frac{3}{4}\right)\left(\frac{1}{M_{10}''}\right) e^{-i\varepsilon_{10}''} \left\{1 - i\eta\left(\frac{\Delta E}{2} + \frac{\Delta\sigma}{3}\right)\right\}^2 \tag{4-20}$$

Expanding the bracketed term according to the binomial expansion and neglecting second-order terms, we have

$$\left\{ 1 - in\left(\frac{\Delta E}{2} + \frac{\Delta \sigma}{3} \right) \right\}^2 = 1 - 2in\left(\frac{\Delta E}{2} + \frac{\Delta \sigma}{3} \right)$$

Thus equation 4-20 reduces to the form

$$\frac{c_o^2}{c^2} = \frac{3}{4}\left(\frac{1}{M''_{10}} \right) e^{-i\varepsilon''_{10}} \left\{ 1 - in\left(\Delta E + \frac{2}{3}\Delta\sigma \right) \right\} \tag{6-48}$$

Combining equations 6-43 and 6-48 we obtain

$$\frac{2\xi}{R} = \frac{3}{4}\left(\frac{p}{\rho_o c_o^2} \right) \left\{ 1 - in\left(\Delta E + \frac{2}{3}\Delta\sigma \right) \right\} \tag{6-49}$$

From equation 6-49 we note that if the radial expansion, ξ, lags behind the pressure, p, internal damping must be present. Moreover, if the internal damping is of the simple form, as described by equation 6-49, then the amount of phase-lag in any harmonic will be roughly proportional to the frequency n.

If the phase-lag is large, then equation 6-49 will not be sufficiently accurate and we must use the exact form of $(\frac{c0}{c})^2$ for substitution in equation 6-49. This exact form from section IV is

$$\left(\frac{c_o}{c}\right)^2 = \left(\frac{1-\sigma^2}{1-F_{in}} \right) \left[\frac{1}{1-\sigma^2} - \frac{\sigma^2(1+in\Delta\sigma)^2}{1-\sigma^2} \right] \left[\frac{1}{1+in\Delta E} \right]$$

where by analogy with equation 2-89, for the elastic tube we write

$$\frac{1}{1-F_{10}} = \frac{e^{-i\varepsilon''_{10}}}{M''_{10}}$$

Thus the exact form for $(\frac{c_0}{c})^2$, with $\sigma = 1/2$, is

$$\left(\frac{c_0}{c}\right)^2 = \left(\frac{3}{4}\right)\frac{e^{-i\varepsilon''_{10}}}{M''_{10}}\left[\frac{4}{3} - \frac{1}{3}\left(1 + 2in\Delta\sigma - n^2(\Delta\sigma)^2\right)\right]\left(\frac{1}{1+in\Delta E}\right)$$

Substituting this representation for $(\frac{c_0}{c})^2$ in equation 6-43, we find that

$$\frac{2\xi}{R} = \frac{p}{\rho_0 c_0^2}\left(\frac{c_0^2}{c^2}\right)M''_{10}\,e^{i\varepsilon''_{10}} \qquad (6\text{-}43)$$

$$= \frac{3}{4}\left(\frac{p}{\rho_0 c_0^2}\right)\left[\frac{4}{3} - \frac{1}{3}\left(1 + 2in\Delta\sigma - n^2(\Delta\sigma)^2\right)\right]\left(\frac{1}{1+in\Delta E}\right)$$

$$= \frac{3}{4}\left(\frac{p}{\rho_0 c_0^2}\right)\left[1 - \frac{2}{3}in\Delta\sigma + \frac{1}{3}(n\Delta\sigma)^2\right]\left(\frac{1}{1+in\Delta E}\right)$$

(6-50)

From equation 6-50 we note that

$$\left(\frac{2}{R}\right) \text{phase}\{\xi\} = \left(\frac{3}{4}\right)\left(\frac{1}{\rho_0 C_0^2}\right) \text{phase}\{p\}$$

$$+ \text{phase}\left\{1 + \frac{1}{3}(n\Delta\sigma)^2 - i\,\frac{2}{3}(n\Delta\sigma)\right\} - \text{phase}\{1 + i n\Delta E\}$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{\rho_0 C_0^2}\right) \text{phase}\{p\} + \tan^{-1} \frac{\left(-\frac{2}{3} n\Delta\sigma\right)}{1 + \frac{1}{3}(n\Delta\sigma)^2} - \tan^{-1} n\Delta E$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{\rho_0 C_0^2}\right) \text{phase}\{p\} - \tan^{-1} \frac{2n\Delta\sigma}{3 + (n\Delta\sigma)^2} - \tan^{-1} n\Delta E$$

Thus, the phase lag of ξ behind p is of the amount

$$\tan^{-1} n\Delta E + \tan^{-1} \frac{2n\Delta\sigma}{3 + (n\Delta\sigma)^2}$$

From this we may obtain estimates of ΔE and $\Delta\sigma$ by combining the results from several harmonics.

Experimental verification of equation 6-47 can be had by referring to the results obtained by Lawton and Greene (1956). They succeeded in obtaining measurements of variation in diameter throughout the pulse cycle by filming the motion of very small beads sewn to the abdominal aorta of the dog. Two typical results (at T = 33/120 = 0.275 sec and T = 0.352 sec) for the variation in diameter together with the corresponding variation in pressure are shown in figures 38 and 39. The results of Fourier analysis up to

the fourth harmonic are given in table III. These results are indicated in modulus and phase form, i.e., in the form $M_m \cos (mnt - \phi_m)$, n being the order of the harmonic. These results show no steady increase in phase lag with respect to frequency. The results of curve #1 seem to show a decrease in phase lag with frequency. This may be illusory. The amplitudes of the third and fourth harmonics are small, that of the third harmonic being less than one-sixth of that of the fundamental, that of the fourth harmonic about 5%. Thus the estimate of the phase lag cannot be expected to be very accurate. It seems reasonable to conclude from these results that, although there are irregular variations in phase between pressure and diameter, these variations are not inconsistent with the assumption that there is no damping in the wall. Therefore, until measurements of greater accuracy become available, the simple form of the theory, i.e., $k \to -\infty$, may be considered to be reasonably accurate.

TABLE III

Values of Fourier Coefficients of the First Four Harmonics for the Pressure and Diameter Variations Shown in Figures 38 and 39

	HARMONIC	PRESSURE		DIAMETER		PHASE-LAG (DEGREES)
		M_m	ϕ_m	$M_m \times 10^{-3}$	ϕ_m	
Curve #1						
	1	18.74	75.67	12.17	86.31	10.64
	2	6.80	128.67	4.05	133.55	4.88
	3	3.14	154.45	2.14	149.68	-4.77
	4	1.56	156.75	0.50	110.03	-46.72
	CONSTANT TERM	69.73		1.32		
Curve #2						
	1		62.35		56.85	5.50
	2		116.50		117.05	-0.55
	3		152.85		136.98	15.87
	4		124.50		122.60	1.90

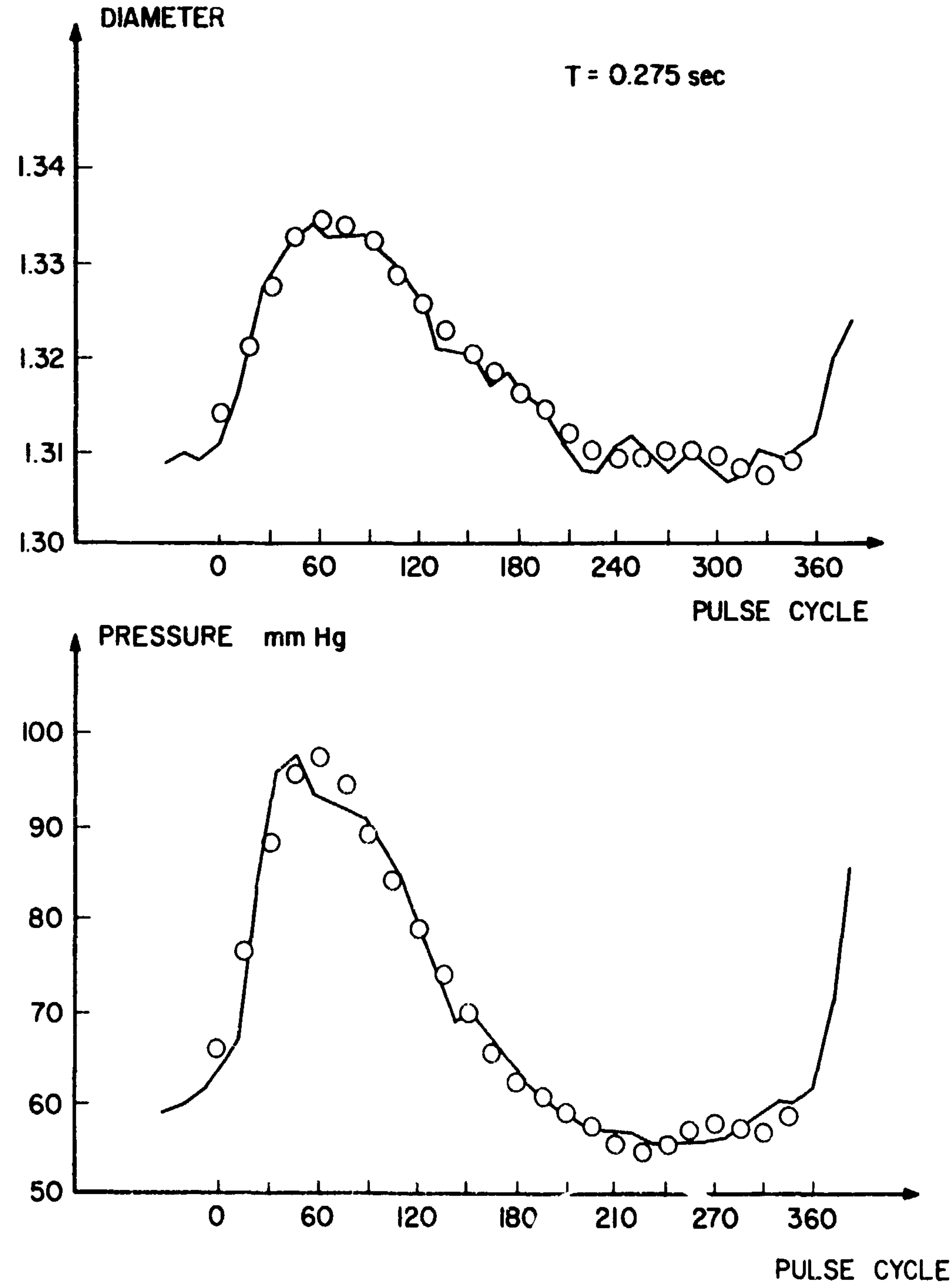

Figure 38. Cyclic variation in diameter of the abdominal aorta of the dog with respect to the pulse cycle. The observed points are joined by straight lines. The circles are points on a four-harmonic Fourier series fitted to the observations. The corresponding variation of pressure with respect to the pulse cycle is also shown.

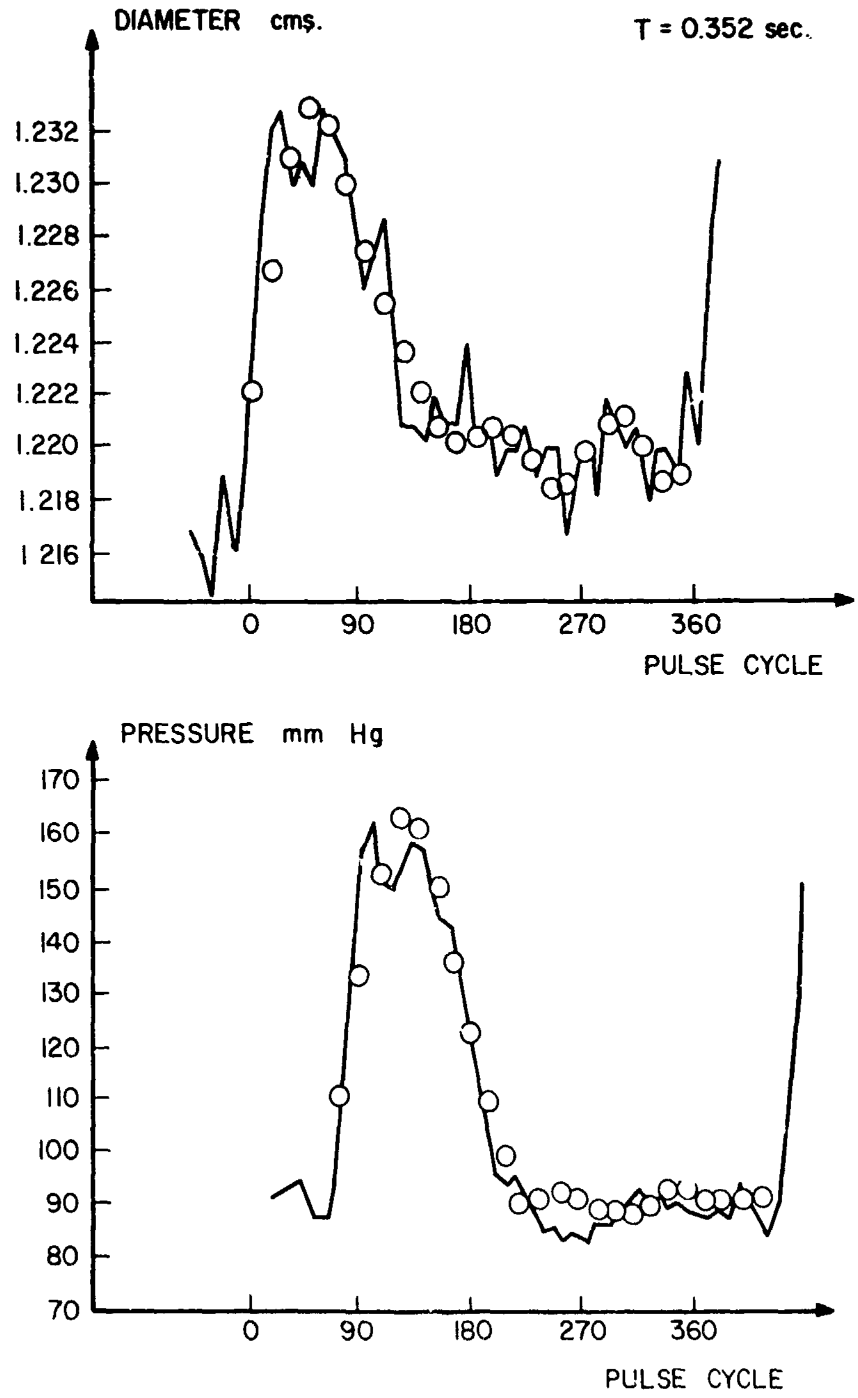

Figure 39. Cyclic variation in diameter of the abdominal aorta of the dog with respect to the pulse cycle. The observed points are joined by straight lines. The circles are points on a four-harmonic Fourier series fitted to the observations. The corresponding variation of pressure with respect to the pulse cycle is also shown.

SECTION VII

JUNCTIONS AND DISCONTINUITIES

From experimental observations it is quite clear that the arterial pulse-wave increases in amplitude and develops secondary waves as it travels down the arterial tree. See figure 40. With this in view, we shall now develop a more accurate representation of the amount of damping of the arterial pulse-wave as it travels from the heart to the peripheries. To this end, we shall consider the traveling pulse-wave in terms of its harmonic components and include the presence of wave reflection at arterial junctions and discontinuities. An approximate method will be indicated for estimating a reflection coefficient as a function of the area ratio of the branches to the parent tube. This coefficient will be used in estimating the reflections produced at the iliac and coeliac junctions and at discontinuities introduced by insertion of an electromagnetic flowmeter as is required in some methods for measuring pulsatile blood flow.

THE REFLECTION COEFFICIENT

Corresponding to equation 6-18, relating the fluid pressure to the average longitudinal fluid velocity, we write

$$\bar{w}_{1\,\text{ELASTIC}} = \frac{A_1}{\rho_0 c} M''_{10}(\alpha)\, e^{i\epsilon''_{10}(\alpha)} \qquad (7\text{-}1)$$

If the viscosity of the fluid approaches zero, then $\alpha^2 = \frac{R^2 n}{\nu} \to \infty$ and

$$M''_{10}(\alpha) = |1 + \eta F_{10}(\alpha)| \to 1$$

since

$$F_{10}(\alpha) = \frac{2 J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)} \longrightarrow 0 \quad \text{AS} \quad \alpha \to \infty,$$

and $\eta = \frac{C_1}{A_1/\rho_0 c}$ remains finite.

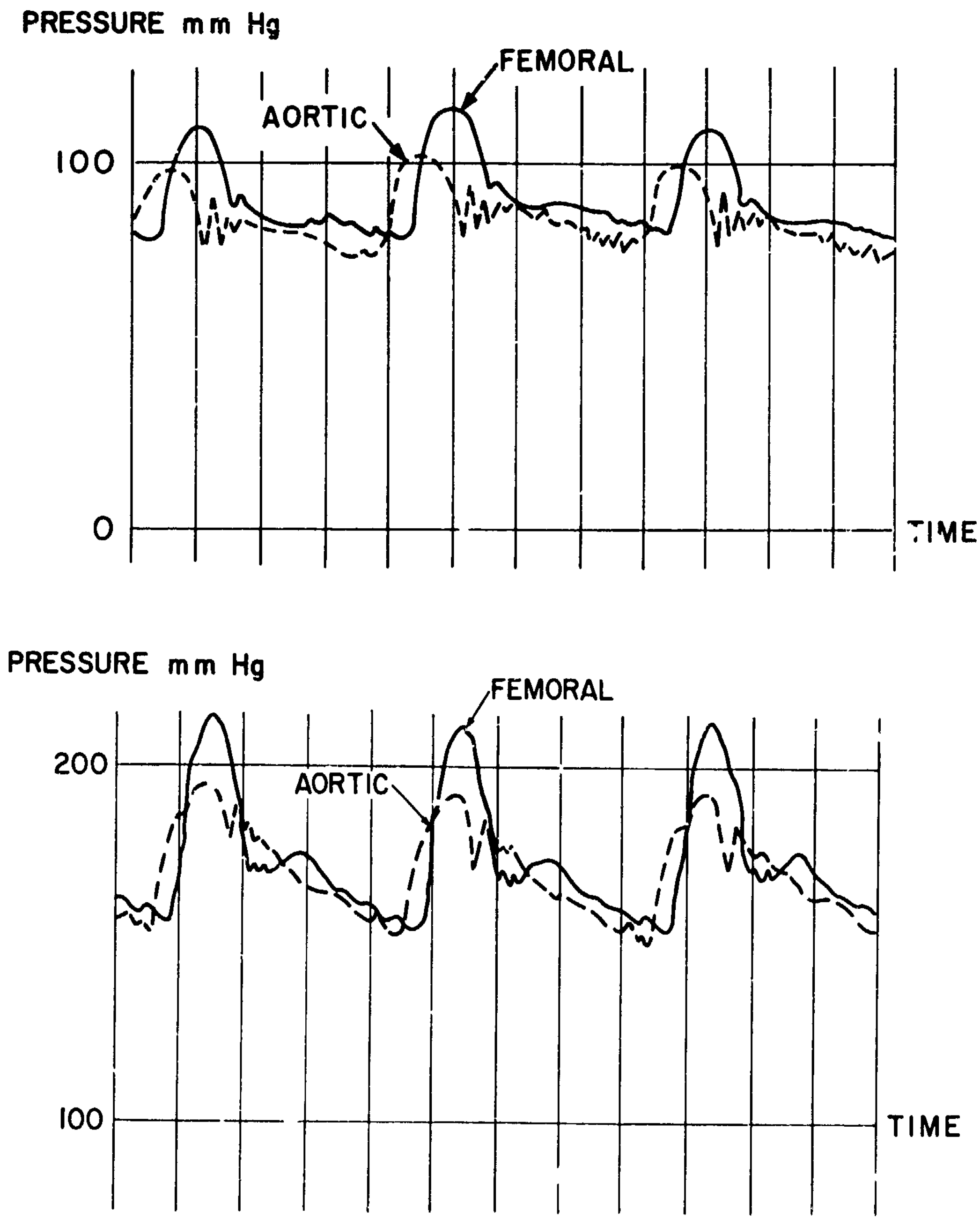

Figure 40. Rise in the peak of the aortic and femoral pulse-pressure during transmission in the dog. (By courtesy of Dr. R. W. Stacy)

Moreover, $\varepsilon''_{10}(\alpha)$ = phase $\{1 + \eta F_{10}(\alpha)\} = 0$. Thus, if we assume that the fluid is inviscid, then $M''_{10}(\alpha) = 1$ and $\varepsilon''_{10}(\alpha) = 0$ and equation 7-1 reduces to the form

$$\bar{w}_{1\,\text{elastic}} = \frac{A}{\rho_0 c} \tag{7-2}$$

We will now consider the error that is generated when we use the approximate form of the pressure-velocity relation (equation 7-2) instead of equation 7-1.

Suppose there is a sudden reduction in the size of an artery from a fixed radius R to a fixed radius r. See figure 41. Let α_1 and α_2 be the values of α in the larger and smaller tube respectively. From the relation $\alpha^2 = R^2 n/\nu$, we note that the value of α is directly proportional to the radius and, since $r < R$, it follows that $\alpha_2 < \alpha_1$.

We will assume that on account of a change in the tube diameter, there is partial transmission and partial reflection of the incident wave. It is convenient to have the incident wave traveling to the right and the origin of the longitudinal axis of the tube located at the junction. To allow for a possible change of phase, we use the complex exponential rather than the sine or cosine. The incident pressure wave traveling to the right is represented by

$$A_1\, e^{in(t - z/c_1)}$$

The reflected wave traveling to the left is denoted by

$$A_4\, e^{in(t + z/c_1)}$$

The transmitted pressure wave travels in the positive direction in the smaller tube and can be represented by

$$A_2\, e^{in(t - z/c_2)}$$

Note that c_1 and c_2 are the wave velocities in the large and small tubes respectively.

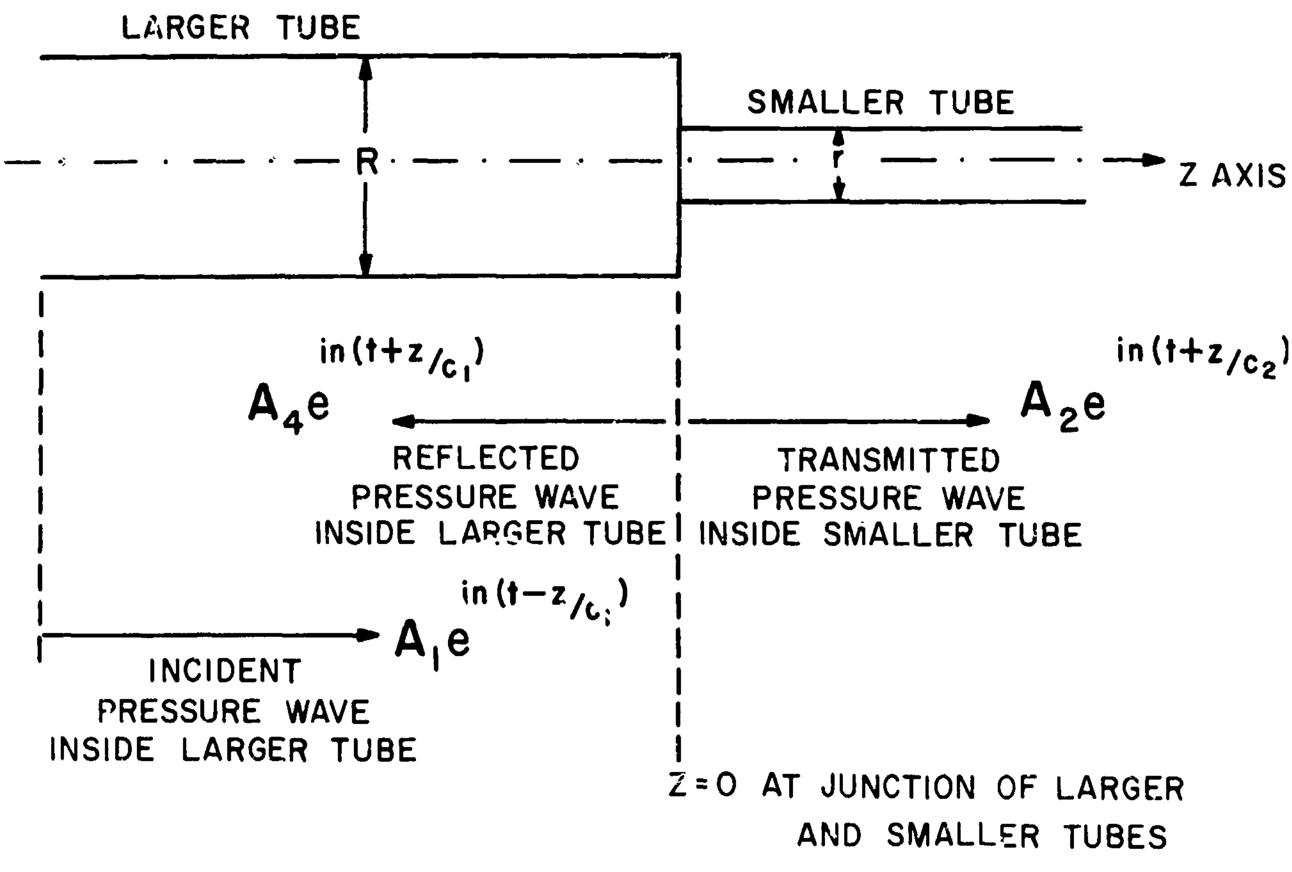

Figure 41. The incident, reflected and transmitted wave at the junction of a larger and smaller tube.

Clearly, the magnitude of the total pressure wave on the left-hand side of the junction is

$$A_1 e^{in(t - z/c_1)} + A_4 e^{in(t + z/c_1)}$$

On the right-hand side of the junction, the magnitude of the pressure wave is

$$A_2 e^{in(t - z/c_2)}$$

Now we shall use two conditions which exist at the junction z=0. These are:

(1) The pressure is the same on both sides of the junction at z=0.
(2) The volume rate of flow is the same on both sides of the junction at z=0.

Imposing these two conditions in turn, we note that since the pressure has to be the same on both sides of the junction at the point z=0, we write

pressure on the left of junction = pressure on the right of junction

i.e., pressure due to incident wave + pressure propagated by reflected wave = pressure due to transmitted wave

i.e., $$A_1 e^{in(t - z/c_1)} + A_4 e^{in(t + z/c_1)} = A_2 e^{in(t - z/c_2)} \tag{7-3}$$

At the point of discontinuity, z=0, we have from equation 7-3

$$A_1 e^{int} + A_4 e^{int} = A_2 e^{int}$$

or $$A_1 + A_4 = A_2 \tag{7-4}$$

Now we shall consider the elastic tube in the limiting condition of stiff constraint. From equation 6-11 we may write the magnitude of the average longitudinal fluid velocities in the large and small tubes as

$$\bar{w}_1\Big]_{\text{LARGE TUBE}} = \frac{\sqrt{3}}{2}\left(\frac{A_1 - A_4}{\rho_0 c_0}\right)\left[M'_{10}(\alpha_1)\right]^{1/2} e^{\frac{i\varepsilon'_{10}(\alpha_1)}{2}} \tag{7-5}$$

$$\bar{w}_1\Big]_{\text{SMALL TUBE}} = \frac{\sqrt{3}}{2}\left(\frac{A_2}{\rho_0 c'_0}\right)\left[M'_{1}(\alpha_2)\right]^{1/2} e^{\frac{i\varepsilon'_{10}(\alpha_2)}{2}} \tag{7-6}$$

In equation 7-5 for the larger tube, the magnitude of the pressure is $(A_1 - A_4)$. This follows from the fact that when considering volume rate of flow across the junction z=0, there is a direction involved. For this reason we must take the difference in the magnitude of the incident and reflected pressure waves. Note also that c_0 and c_0' are the limiting velocities of the wave propagated in the two tubes for $\alpha = (R^2 n/\nu) \rightarrow \infty$, i.e., for liquids of very small viscosity. For continuity of flow across the junction, we equate the volume rate of flow on both sides and write

$$Q]_{\text{large tube}} = Q]_{\text{small tube}}$$

i.e.,

$$(\pi R^2)\frac{\sqrt{3}}{2}\left(\frac{A_1 - A_4}{\rho_0 c_0}\right)\left[M_{10}'(\alpha_1)\right]^{1/2} e^{\frac{i\epsilon_{10}'(\alpha_1)}{2}}$$

$$= (\pi r^2)\frac{\sqrt{3}}{2}\left(\frac{A_2}{\rho_0 c_0'}\right)\left[M_{10}'(\alpha_2)\right]^{1/2} e^{\frac{i\epsilon_{10}'(\alpha_2)}{2}} \qquad (7\text{-}7)$$

or

$$R^2\left(\frac{A_1 - A_4}{\rho_0 c_0}\right)\left[M_{10}'(\alpha_1)\right]^{1/2} e^{\frac{i\epsilon_{10}'(\alpha)}{2}} = r^2\left(\frac{A_2}{\rho_0 c_0'}\right)\left[M_{10}'(\alpha_2)\right]^{1/2} e^{\frac{i\epsilon_{10}'(\alpha_2)}{2}} \qquad (7\text{-}8)$$

According to the Moens-Korteweg formula,

$$c_0 = \left(\frac{hE}{2R\rho_0}\right)^{1/2}$$

we will assume that in the larger tube with fixed radius R,

$$c_0 \propto \frac{1}{R^{1/2}}$$

and in the smaller tube with fixed radius r,

$$c_0' \propto \frac{1}{r^{1/2}}$$

Considering the thickness, h, and the modulus of elasticity, E, to remain the same in the larger and smaller tubes, this assumption implies that the mass loading on the two tubes is the same. With this assumption, equation 7-8 assumes the form

$$R^2\left(\frac{A_1 - A_4}{\rho_0}\right)R^{1/2}\left[M_{10}'(\alpha_1)\right]^{1/2} e^{\frac{i\varepsilon_{10}'(\alpha_1)}{2}}$$

$$= r^2\left(\frac{A_2}{\rho_r}\right)r^{1/2}\left[M_{10}'(\alpha_2)\right]^{1/2} e^{\frac{i\varepsilon_{10}'(\alpha_2)}{2}}$$

(7-9)

Equation 7-9 may be written in the form

$$\frac{A_1 - A_4}{A_2} = \left(\frac{r}{R}\right)^{2.5}\left[\frac{M_{10}'(\alpha_2)}{M_{10}'(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon_{10}'(\alpha_2) - \varepsilon_{10}'(\alpha_1)\right]}$$

(7-10)

For convenience, we denote the right-hand side of equation 7-10 by λ and write

$$\frac{A_1 - A_4}{A_2} = \lambda \tag{7-11}$$

Now we need an expression in terms of λ for the ratio of the reflected pressure wave to the incident pressure wave. A_4/A_1. This ratio is known as the reflection coefficient for the junction. From equation 7-11 we may write the following.

$$\frac{A_2 - (A_1 - A_4)}{A_2} = 1 - \lambda \tag{7-12}$$

$$\frac{A_2 + (A_1 - A_4)}{A_2} = 1 + \lambda \tag{7-13}$$

Taking the ratio of the left and right sides of equations 7-12 and 7-13, we have

$$\frac{A_2 - A_1 + A_4}{A_2 + A_1 - A_4} = \frac{1 - \lambda}{1 + \lambda}$$

or

$$\frac{(A_2 - A_1) + A_4}{(A_2 - A_4) + A_1} = \frac{1 - \lambda}{1 + \lambda} \tag{7-14}$$

Now from equation 7-4 we may write

$$A_2 - A_1 = A_4$$

and

$$A_2 - A_4 = A_1$$

Therefore equation 7-14 becomes

$$\frac{A_4 + A_4}{A_1 + A_1} = \frac{1 - \lambda}{1 + \lambda}$$

or

$$\frac{A_4}{A_1} = \frac{1 - \lambda}{1 + \lambda} \tag{7-15}$$

The modulus of the complex quantity A_4/A_1 denotes the ratio of the amplitude of the reflected wave to the incident wave. Its phase is

$$\text{phase}\left(\frac{A_4}{A_1}\right) = \text{phase}\ (A_4) - \text{phase}\ (A_1)$$

which denotes the change in phase of the incident wave upon reflection at the junction.

The ratio A_2/A_1 is known as the transmission coefficient. We obtain the value of A_2/A_1 in terms of λ as follows. We know that

$$\frac{A_1 - A_4}{A_2} = \lambda$$

or

$$\frac{A_2}{A_1 - A_4} = \frac{1}{\lambda}$$

or

$$A_2 = \frac{1}{\lambda}(A_1 - A_4)$$

or

$$\frac{A_2}{A_1} = \frac{1}{\lambda}\left(\frac{A_1 - A_4}{A_1}\right)$$

From the expression for the reflection coefficient, we know that

$$\frac{A_4}{A_1} = \frac{1 - \lambda}{1 + \lambda}$$

Thus

$$\frac{A_1}{A_4} = \frac{1 + \lambda}{1 - \lambda}$$

or

$$\frac{A_1 - A_4}{A_1} = \frac{(1 + \lambda) - (1 - \lambda)}{1 + \lambda} = \frac{2\lambda}{1 + \lambda}$$

Therefore

$$\frac{A_2}{A_1} = \frac{1}{\lambda}\left(\frac{A_1 - A_4}{A_1}\right) = \frac{1}{\lambda}\left(\frac{2\lambda}{1 + \lambda}\right) = \frac{2}{1 + \lambda}$$

Note that if we had used the simplified form of the pressure velocity relation as described by equation 7-2, then equation 7-10 would have the form

$$\frac{A_1 - A_4}{A_2} = \left(\frac{r}{R}\right)^{2.5} = \lambda \qquad (7\text{-}16)$$

Equation 7-16 may be obtained from equation 7-10 if $M'_{10}(\alpha_1) = M'_{10}(\alpha_2)$ and $\varepsilon'_{10}(\alpha_1) = \varepsilon'_{10}(\alpha_2)$, i.e., if the viscosity of the fluid is neglected.

We may therefore look upon the factor

$$\left[\frac{M'_{10}(\alpha_2)}{M'_{10}(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon'_{10}(\alpha_2) - \varepsilon'_{10}(\alpha_1)\right]}$$

appearing in equation 7-10 as a "throttling" effect due to the fluid viscosity.

The above equations can be used for the division of an artery into a number of branches of equal size. All that is necessary is to multiply the right-hand side of equation 7-10 by the number of branches in order to account for the larger amount of flow. See equation 7-7. Thus, for a division into two equal branches, we have from equation 7-10

$$\lambda = 2\left(\frac{r}{R}\right)^{2.5}\left[\frac{M'_{10}(\alpha_2)}{M'_{10}(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon'_{10}(\alpha_2) - \varepsilon'_{10}(\alpha_1)\right]} \tag{7-17}$$

Equation 7-17 applies to the constrained tube.

Next we consider the artery as an elastic tube with equal velocities of wave propagation on both sides of the junction, $c_0 = c_0'$, and obtain an expression for λ. From the condition of continuity of the volume rate of flow across the junction we write

$$(\pi R^2)\left(\frac{A_1 - A_4}{\rho_0 c_0}\right)\left[M''_{10}(\alpha_1)\right] e^{i\varepsilon''_{10}(\alpha_1)}$$

$$= (\pi r^2)\left(\frac{A_2}{\rho_0 c_0'}\right)\left[M''_{10}(\alpha_2)\right] e^{i\varepsilon''_{10}(\alpha_2)} \tag{7-18}$$

Now if $c_0 = c_0'$, then equation 7-18 may be written as

$$\lambda_{ELASTIC} = \frac{A_1 - A_4}{A_2} = \left(\frac{r}{R}\right)^2 \left[\frac{M''_{10}(\alpha_2)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_2) - \varepsilon''_{10}(\alpha_1)\right]} \qquad (7\text{-}19)$$

For a division of an artery, considered as an elastic tube, into two equal branches we multiply the right-hand side of equation 7-19 by 2 to obtain:

$$\lambda_{ELASTIC} = 2\left(\frac{r}{R}\right)^2 \left[\frac{M''_{10}(\alpha_2)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_2) - \varepsilon''_{10}(\alpha_1)\right]} \qquad (7\text{-}20)$$

The quantity $2\left(\frac{r^2}{R^2}\right)$ in equation 7-20 is the ratio of the combined area of the branches to the area of the original tube and is called the area ratio of the junction. This quantity has been chosen as the abscissa in figures 42-47.

In figure 42 the variation of the amplitude of the reflection coefficient λ with respect to the area ratio, as described by equation 7-17, is indicated for four values of α in the incident tube. Recall that equation 7-17 was established for a tube with stiff longitudinal constraint and equal mass-loading on the original artery and branches. Figure 43 shows the corresponding variation of phase lag of the reflected wave. Figures 44 and 45 are similar sets of curves with λ as defined by equation 7-20, i.e., for an unconstrained tube, with k=0, σ=1/2 and the wave velocities in the original tube and branches being assumed to be equal. In figures 46 and 47 the mass-loading on the branches has been increased, the assumption being made that the wave velocity is inversely proportional to the radius of the tube. See equation 7-21.

If in equation 7-7 we assume that the wave velocities vary inversely as the tube radii,

$$c_0 \propto \frac{1}{R} \quad \text{and} \quad c_0' \propto \frac{1}{r}$$

then for a division of an artery, considered as a rigid tube, into two equal branches, the expression for λ is of the form

$$\lambda = 2\left(\frac{r}{R}\right)^3 \left[\frac{M'_{10}(\alpha_2)}{M'_{10}(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon'_{10}(\alpha_2) - \varepsilon'_{10}(\alpha_1)\right]} \tag{7-21}$$

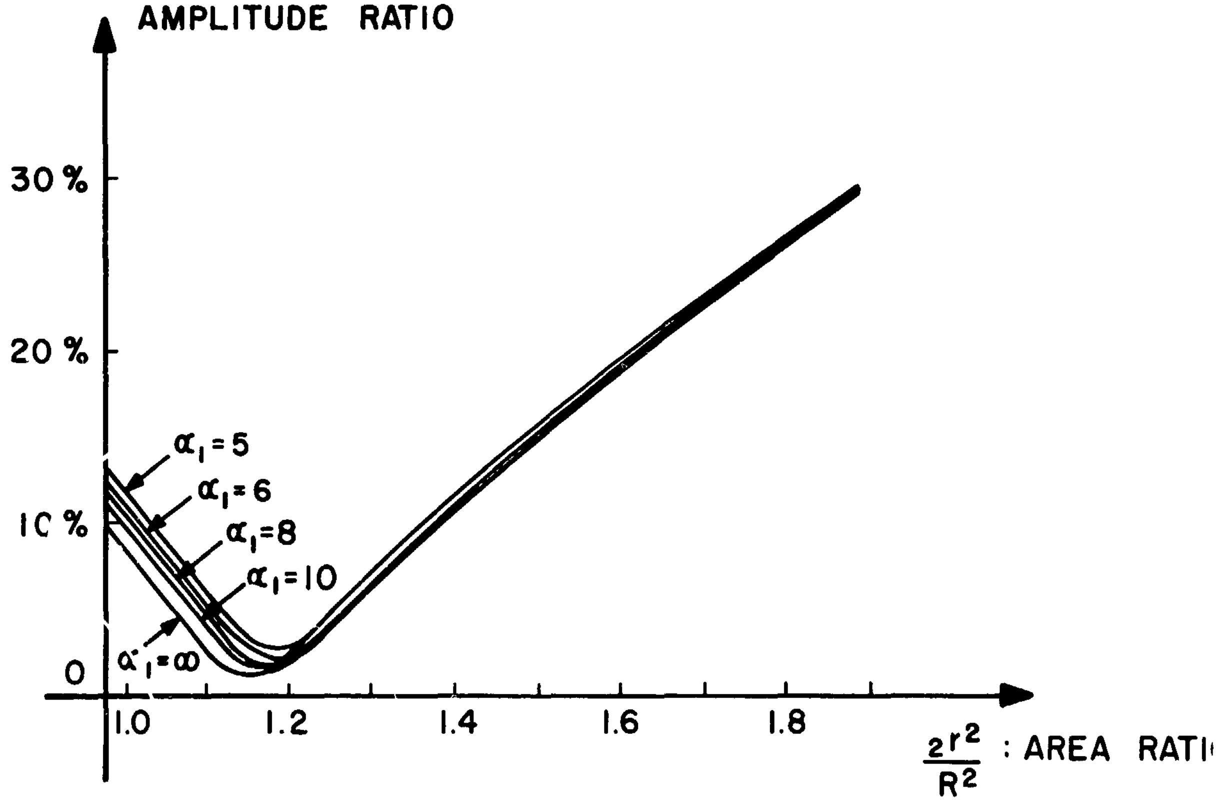

Figure 42. Variation of the amplitude of the reflected wave with respect to the area ratio for four values of α ($\alpha_1 = 5, 6, 8, 10$) in the incident tube. The reflected wave is expressed as a percentage of the incident wave at a division of the artery into two equal branches. The tube is in the condition of limiting longitudinal constraint and filled with a viscous fluid and a nonviscous fluid, $\alpha_1 \to \infty$.

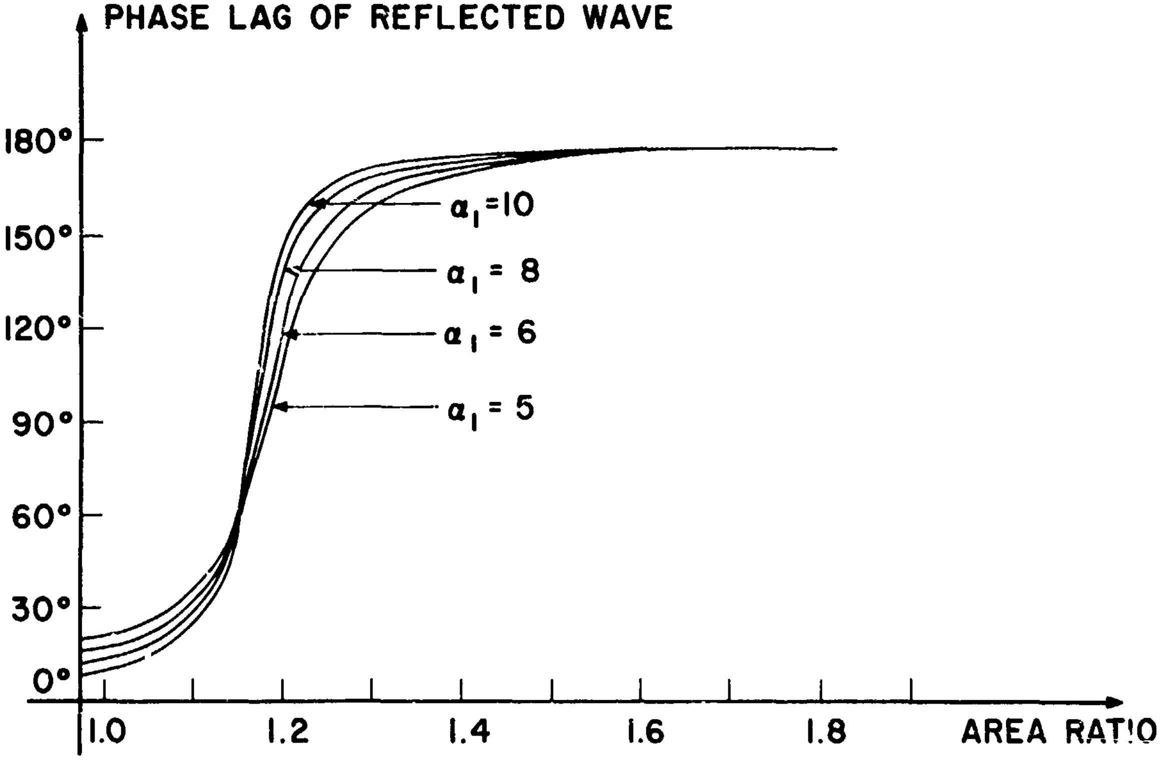

Figure 43. Variation of the phase lag of the reflected wave with respect to the area ratio for the same conditions as in figure 42. For the nonviscous fluid, the phase lag of the reflected wave changes from 0° to 180° at the point where the amplitude ratio is zero.

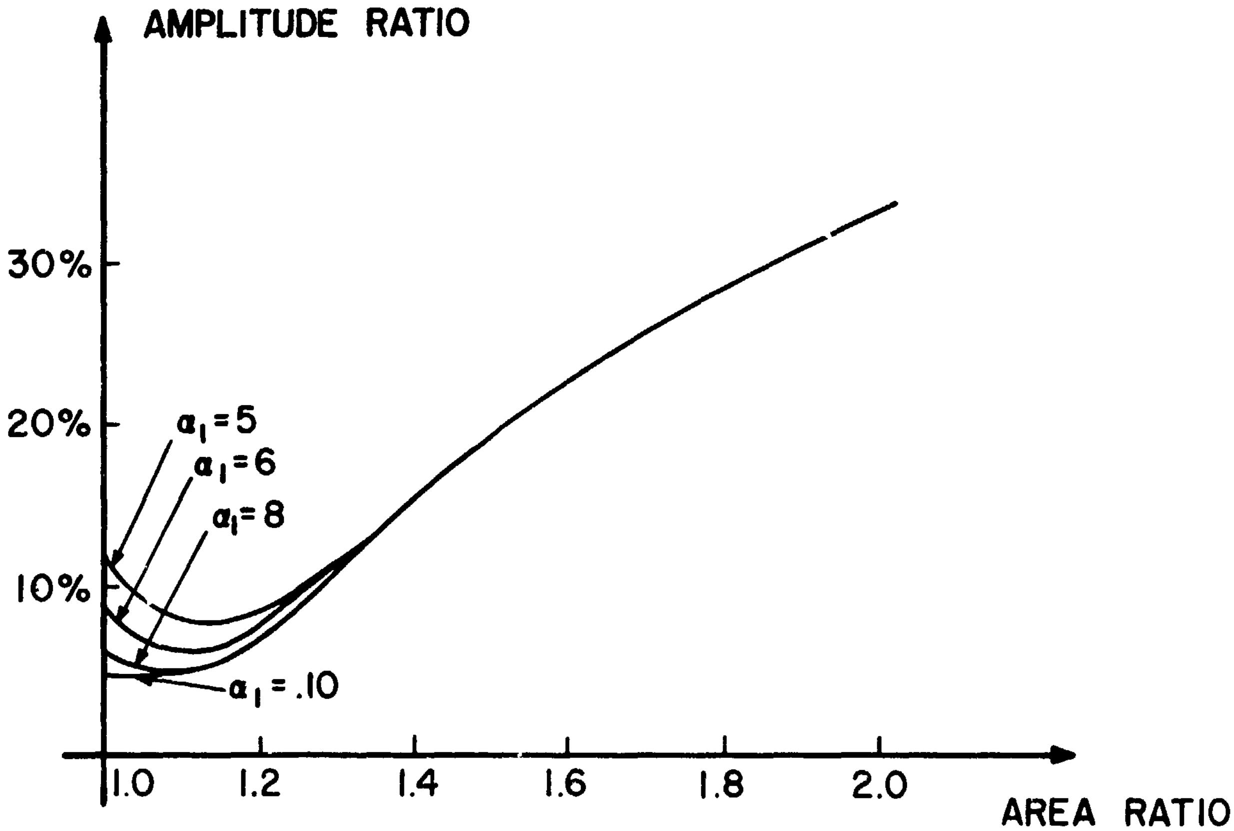

Figure 44. Variation of the magnitude of reflected waves with respect to the area ratio for the freely moving elastic tube, k=0, with the same wave velocity on either side of the junction and σ = 0.5.

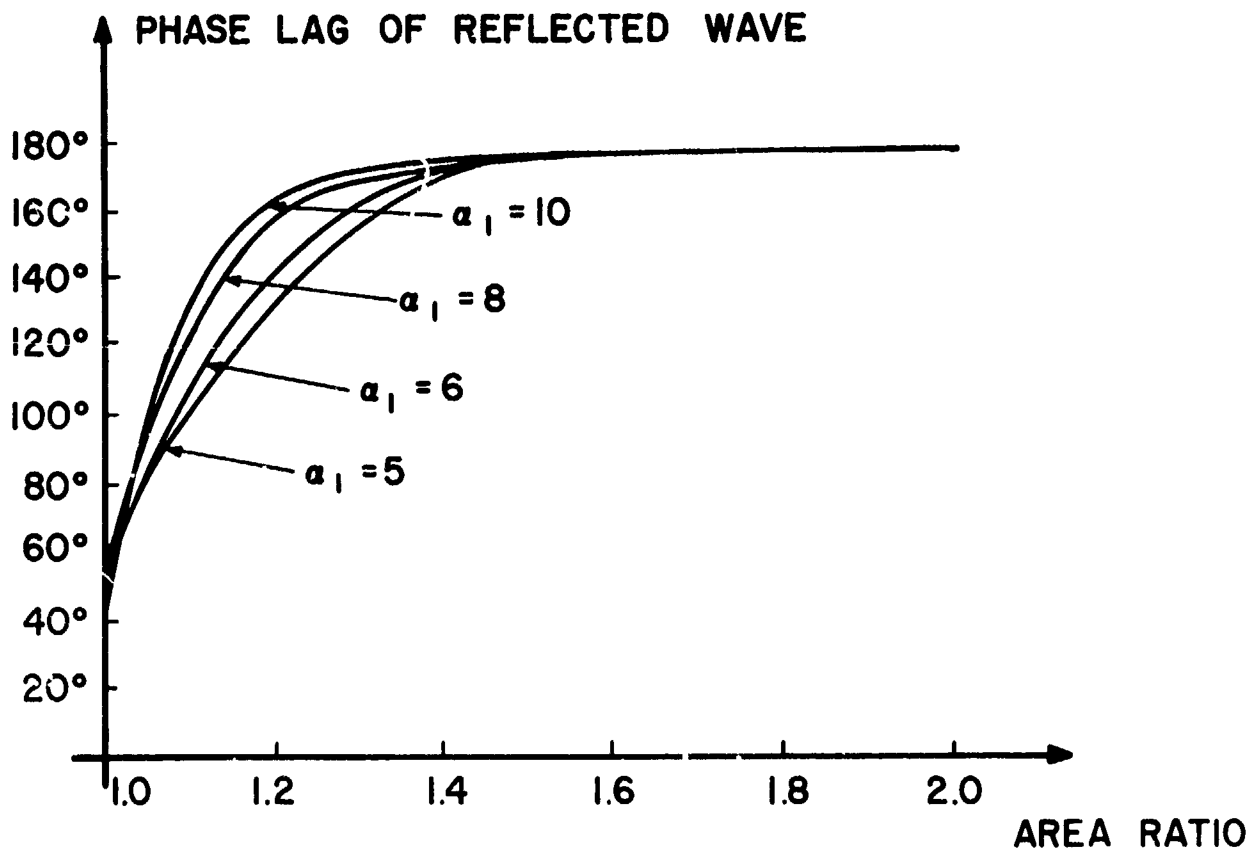

Figure 45. Variation of phase lag of reflected waves for the freely-moving elastic tube with k=0, σ=1/2 and the same wave velocity on either side of the junction.

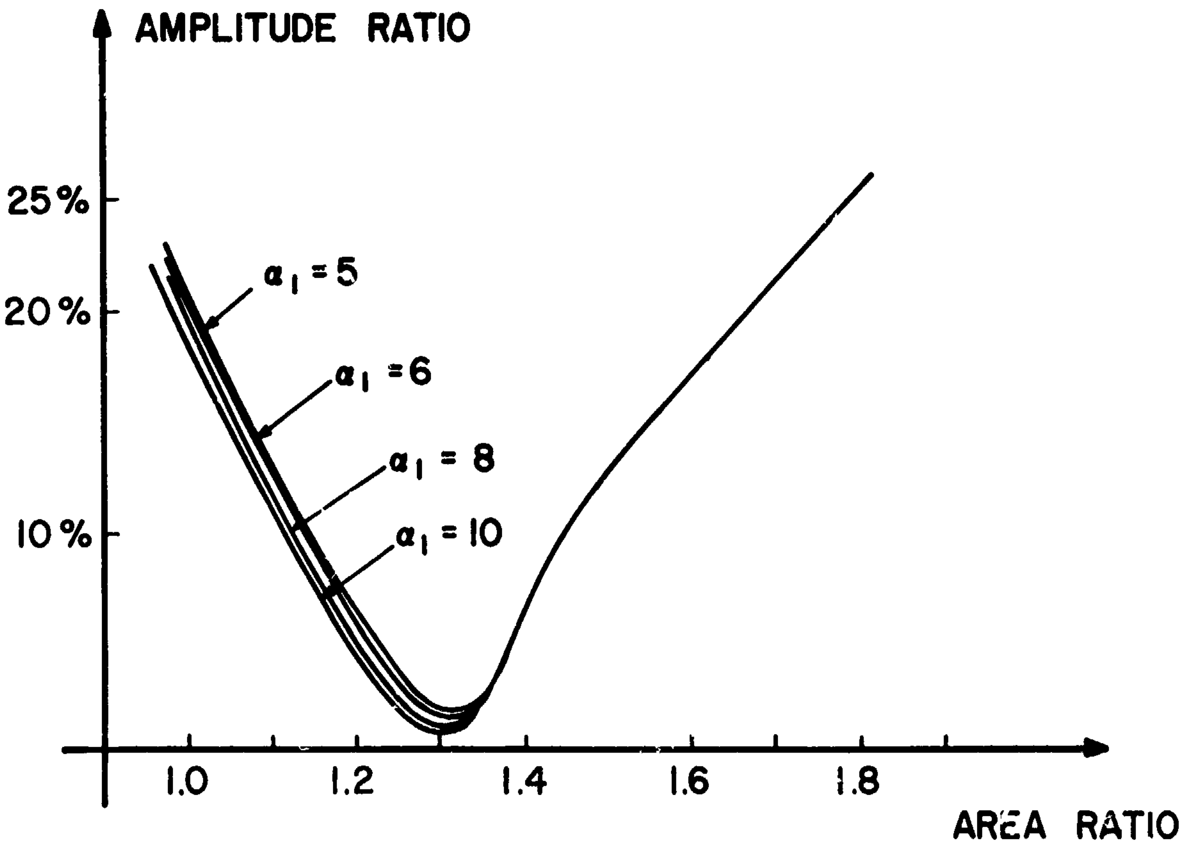

Figure 46. Variation of the magnitude of reflected waves with respect to the area ratio for longitudinally constrained elastic tubes as in figure 42, but with a greater change in wave velocity between incident artery and branches. Here $(c_1/c_2) = (r/R)$.

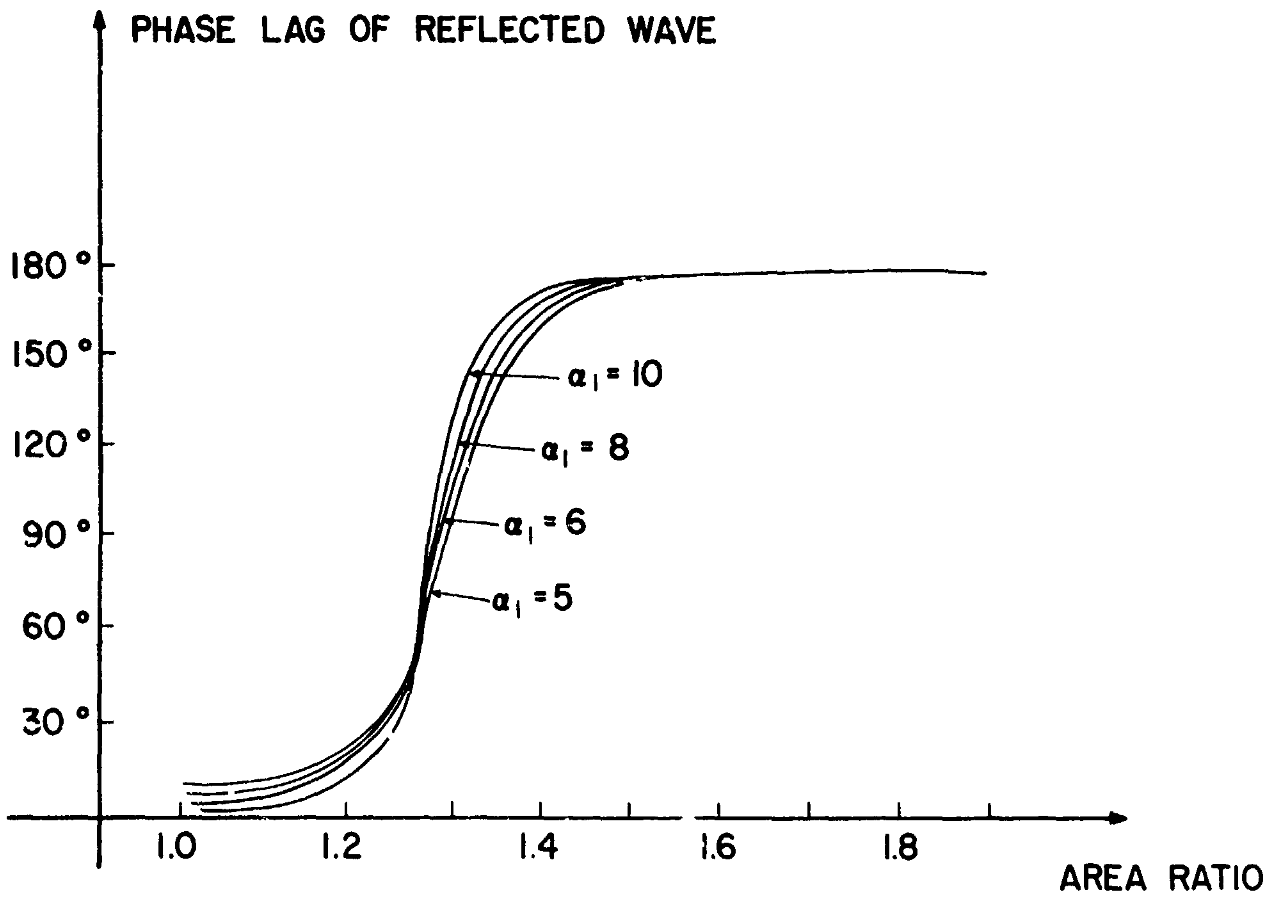

Figure 47. Variation of phase lag of reflected waves with respect to area ratio for the longitudinally constrained elastic tubes as in figure 42, but with greater change in wave velocity between incident artery and branches. Here $(c_1/c_2) = (r/R)$.

Now we shall consider the case when each of the branches is of the same size as the original artery, i.e., r = R. Moreover, if we impose the condition that $\alpha_1 = \alpha_2$, it follows that all the above expressions for λ reduce to the form $\lambda = 2$. From equation 7-15 for the reflection coefficient

$$\frac{A_4}{A_1} = \frac{1 - \lambda}{1 + \lambda} \tag{7-15}$$

it follows that for λ=2

$$\frac{A_4}{A_1} = -\frac{1}{3} \tag{7-22}$$

Equation 7-22 implies that for the conditions imposed, the amplitude of the reflected wave is one-third the amplitude of the incident wave. The negative sign indicates that the incident and reflected waves are 180° of phase.

In figures 42-47, note that all the curves are similar. The curves have a minimum at an area ratio slightly greater than 1.0. This minimum value occurs at a higher value of the area ratio, the greater the difference in velocity between the original artery and the branches. Note also that as this difference in velocity increases, the minimum point on the curve is sharper and lower. In figure 42 the minimum point is always less than 3%, whatever the value of α. Although the minimum is higher for the unconstrained tube, it is less critical.

The change in phase-lag for small variations in the area ratio is very large near the minimum on its lower side. It may be suggested with some confidence, therefore, that if the increase in total area at a division into two branches is of the order of 5%-30%, the amount of reflection will be fairly small, but the change in phase will depend, quite critically, on the particular conditions.

Some enhancement of the harmonic terms in the pressure is, therefore, to be expected at each reflection as long as the increase in total area is not too great. If the area ratio of the junction is greater than about 1.3 or 1.4, the phase-lag will be more than 90°. This would be expected to cause "spreading" rather than "peaking" of the pulse wave.

For three or more branches, the relationships between the reflection coefficient and the area ratio are very similar to those for two branches. The point of minimum reflection is at a higher value of the area ratio, and the minimum amplitude is itself higher. This follows from the greater difference between α_2 and α_1 for the larger number of branches. The change in phase is more gradual, though still rapid on the lower side of the minimum. If the iliac junction in the dog is treated as a division into three equal branches, the reflection coefficient has an amplitude of 14% and the phase change on reflection has a lag of about 40°. Treating the coeliac region as a single complex junction gives an estimated amplitude of 5% and a phase-lag of 65°. The details of the computation are shown in the following subsection.

The above examples are grossly oversimplified to be regarded as directly applicable to reflections in the arterial system as they stand. However, they demonstrate the existence of a condition of minimum reflection, which is optimum from the point of view of impedance matching at the junction and is the condition of maximum energy transfer through it. These examples also indicate the way to a reconciliation of the apparent contradiction between the damping of the pulse wave in transmission and the observed rise in systolic maximum towards the periphery of the arterial system.

When a number of junctions are cascaded in series, the direct method of calculation used in the following section becomes clumsy and tedious, even for a small number of junctions. The presence of other junctions will modify these figures profoundly, since the input impedance of a finite length of tube is not the same as that of a tube of infinite length (Taylor, 1957; Karreman, 1954).

EXAMPLES: THE COELIAC AND ILIAC JUNCTIONS

As an application of the preceding discussion, we will consider the coeliac and iliac junctions in the dog, schematically illustrated in figures 48, 49 and 50, with estimates of the diameter of the arteries measured in cm and of the pulse velocity in cm/sec. For purposes of calculation, shown below, it is not necessary that the values of the diameters and pulse velocities be exact, as long as their ratios are reasonably correct.

First we consider the coeliac junction shown in figure 49. Using equation 7-20, with unequal wave velocities c_0 and c_0' and the fact that

incident flow in = sum of the transmitted flow out through each of the five tubes

we find that

$$\lambda = \frac{A_1 - A_4}{A_2} = \frac{A_1 - A_4}{A_1 + A_4}$$

$$= \sum_{i=2}^{6} \left(\frac{r_i^2}{R^2}\right)\left(\frac{c_0}{c_i}\right)\left[\frac{M_{10}''(\alpha_i)}{M_{10}''(\alpha_1)}\right] e^{i\left[\varepsilon_{10}''(\alpha_i) - \varepsilon_{10}''(\alpha_1)\right]}$$

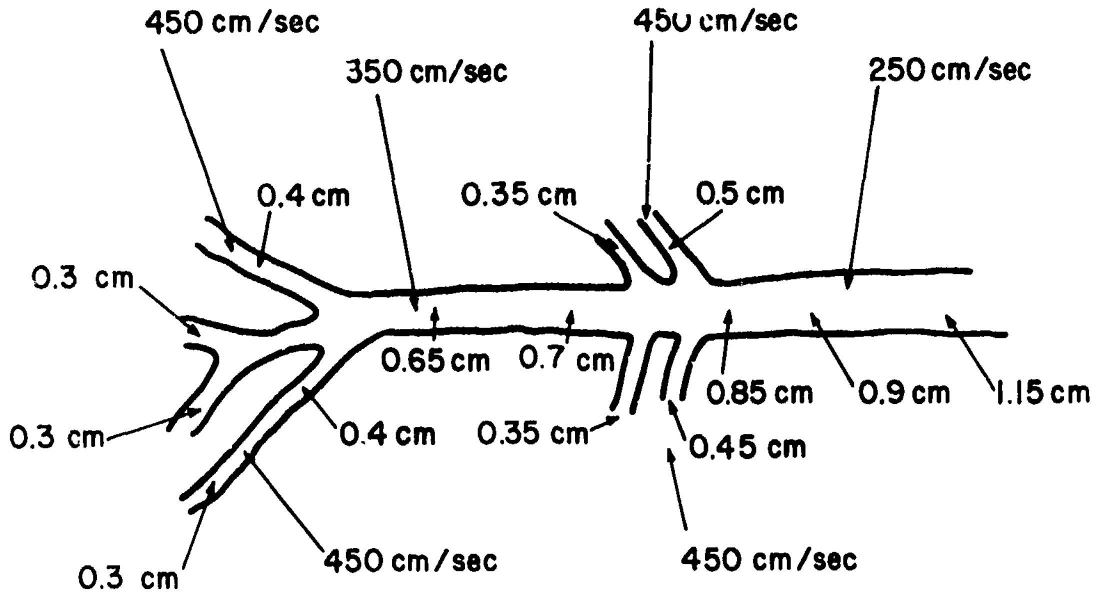

Figure 48. Schematic of part of the arterial system of the dog with estimates of diameter of the arteries in cm and of the pulse velocity in cm/sec.

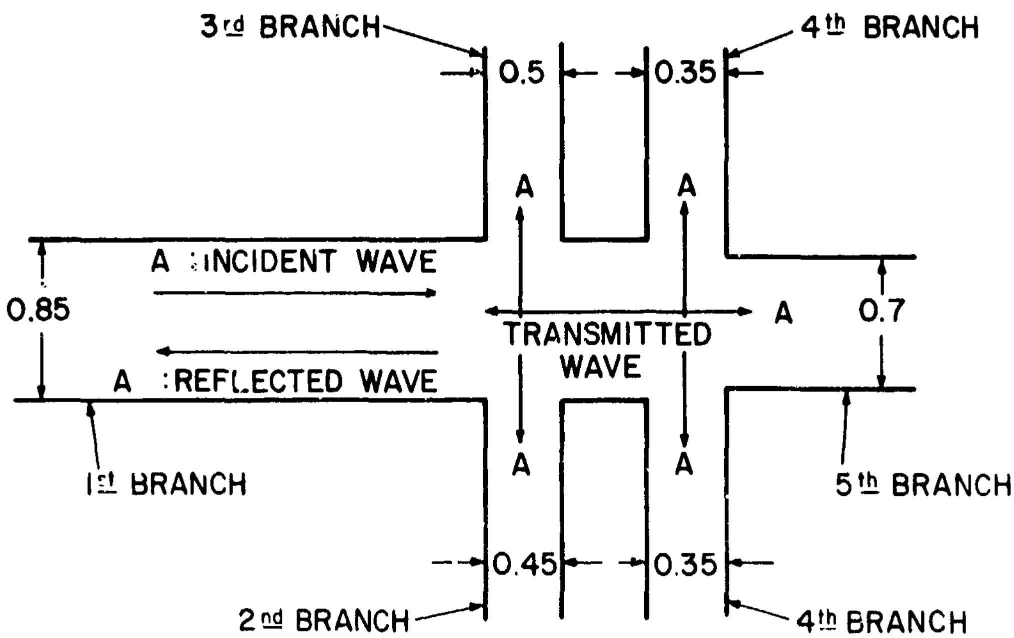

Figure 49. Schematic representing the coeliac junction with diameters in cm and the amplitudes A_1, A_2 and A_4 of the incident, transmitted and reflected waves respectively.

$$= \left(\frac{r_2^2}{R^2}\right)\left(\frac{250}{450}\right)\left[\frac{M''_{10}(\alpha_2)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_2) - \varepsilon''_{10}(\alpha_1)\right]}$$

$$+ \left(\frac{r_3^2}{R^2}\right)\left(\frac{250}{450}\right)\left[\frac{M''_{10}(\alpha_3)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_3) - \varepsilon''_{10}(\alpha_1)\right]}$$

$$+ \left(\frac{r_4^2}{R^2}\right)\left(\frac{250}{450}\right)\left[\frac{M''_{10}(\alpha_4)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_4) - \varepsilon''_{10}(\alpha_1)\right]}$$

$$+ \left(\frac{r_5^2}{R^2}\right)\left(\frac{250}{450}\right)\left[\frac{M''_{10}(\alpha_5)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_5) - \varepsilon''_{10}(\alpha_1)\right]}$$

$$+ \left(\frac{r_6}{R^2}\right)\left(\frac{250}{350}\right)\left[\frac{M''_{10}(\alpha_6)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_6) - \varepsilon''_{10}(\alpha_1)\right]} \qquad (7\text{-}22)$$

Since for the branches numbered four and five we find that

1) the cross sections are the same, $r_4 = r_5$;

2) the pulse velocity is the same, 450 cm/sec;

3) the values of the αs are the same, $\alpha_4 = \alpha_5$;

we may write the sum of the third and fourth terms on the right-hand side of equation 7-22 as

$$2\left(\frac{250}{450}\right)\left(\frac{r_4^2}{R^2}\right)\left[\frac{M''_{10}(\alpha_4)}{M''_{10}(\alpha_1)}\right] e^{i\left[\varepsilon''_{10}(\alpha_4) - \varepsilon''_{10}(\alpha_1)\right]}$$

Thus, from equation 7-22 we write

$$
\begin{aligned}
\lambda = \frac{A_1 - A_4}{A_1 + A_4} &= \left(\frac{250}{450}\right)\left(\frac{r_2^2}{R^2}\right)\left[\frac{M_{10}''(\alpha_2)}{M_{10}''(\alpha_1)}\right] e^{i\left[\varepsilon_{10}''(\alpha_2) - \varepsilon_{10}''(\alpha_1)\right]} \\
&+ \left(\frac{250}{450}\right)\left(\frac{r_3^2}{R^2}\right)\left[\frac{M_{10}''(\alpha_3)}{M_{10}''(\alpha_1)}\right] e^{i\left[\varepsilon_{10}''(\alpha_3) - \varepsilon_{10}''(\alpha_1)\right]} \\
&+ 2\left(\frac{250}{450}\right)\left(\frac{r_4^2}{R^2}\right)\left[\frac{M_{10}''(\alpha_4)}{M_{10}''(\alpha_1)}\right] e^{i\left[\varepsilon_{10}''(\alpha_4) - \varepsilon_{10}''(\alpha_1)\right]} \\
&+ \left(\frac{250}{350}\right)\left(\frac{r_5^2}{R^2}\right)\left[\frac{M_{10}''(\alpha_5)}{M_{10}''(\alpha_1)}\right] e^{i\left[\varepsilon_{10}''(\alpha_5) - \varepsilon_{10}''(\alpha_1)\right]}
\end{aligned}
$$

(7-23)

Note that in equation 7-23 we have denoted the fourth and fifth branches by the fourth branch using subscript 4. For convenience, we have denoted the branch having diameter 0.7 cm as the fifth branch and used subscript 5 (instead of 6) in the last factor on the right-hand side of equation 7-23.

For the calculation of λ we refer to table IV. From this table of values, the four separate terms which make up λ may be calculated first in modulus and phase form, and also in terms of real and imaginary parts as indicated in table V. From table V we write

$$\lambda = (0.9357) + i(0.1181)$$

TABLE IV

TABULAR FORM FOR CALCULATION OF λ

Branch Number	Diameter (cm)	α	M''_{10}	ε''_{10}	$\frac{M''_{10}(\alpha_n)}{M''_{10}(\alpha_1)}$	$\varepsilon''_{10}(\alpha_n) - \varepsilon''_{10}(\alpha_1)$	$\frac{R_n^2}{R_1^2}$
1	0.85	8.5	0.8835	8.56°			
2	0.45	4.5	0.7956	18.83°	0.9005	10.27°	0.2803
3	0.5	5.0	0.8127	16.37°	0.9199	7.81°	0.3460
4	0.35	3.5	0.7462	27.16°	0.8446	18.60°	0.1696
5	0.7	7.9	0.8608	10.78°	0.9743	2.22°	0.6782

TABLE V

TABULAR FORM OF THE FOUR SEPARATE TERMS OF λ IN MODULUS AND PHASE FORM, AND IN REAL AND IMAGINARY PARTS

Term Number	Modulus	Phase	Real Part	Imaginary Part
1	0.1402	10.27°	0.1380	0.0250
2	0.1768	7.81°	0.1752	0.0240
3	0.1592	18.60°	0.1509	0.0508
4	0.4729	2.22°	0.4716	0.0183
			0.9357	0.1181

Thus the reflection coefficient has the value

$$\frac{A_4}{A_1} = \frac{1 - \lambda}{1 + \lambda} = \frac{1 - (0.9357 + i\ 0.1181)}{1 + 0.9357 + i\ 0.1181} = \frac{0.0643 - i\ 0.1181}{1.9357 + i\ 0.1181}$$

It follows that the amplitude ratio has the value

$$\left|\frac{A_4}{A_1}\right| = 0.048$$

and the phase lag is

$$\tan^{-1} \frac{0.1181}{0.0643} + \tan^{-1} \frac{0.1181}{1.9357} \cong 65°$$

These results indicate that for the model of the coeliac junction chosen here and treated as an unconstrained tube, the reflected wave has approximately 5% of the amplitude of the incident wave and is almost 65° behind it in phase.

Similarly, if we consider the iliac junction as a division into three equal branches, as indicated in figure 50, we find that

$$\lambda = 3\left(\frac{0.4}{0.65}\right)^2\left(\frac{350}{450}\right)\left[\frac{M''_{10}(4)}{M''_{10}(6.5)}\right] e^{i\left[\varepsilon''_{10}(4) - \varepsilon''_{10}(6.5)\right]}$$

$$= 0.7910 + i\,(0.1450)$$

Thus $$\left|\frac{A_t}{A_1}\right| = 0.1418$$

and the phase lag is 39.53°.

The preceding results show that for a wide range of conditions, reflection of the pressure wave will cause a moderate increase in amplitude of the transmitted wave for a correspondingly moderate increase in total cross-sectional area at the junction.

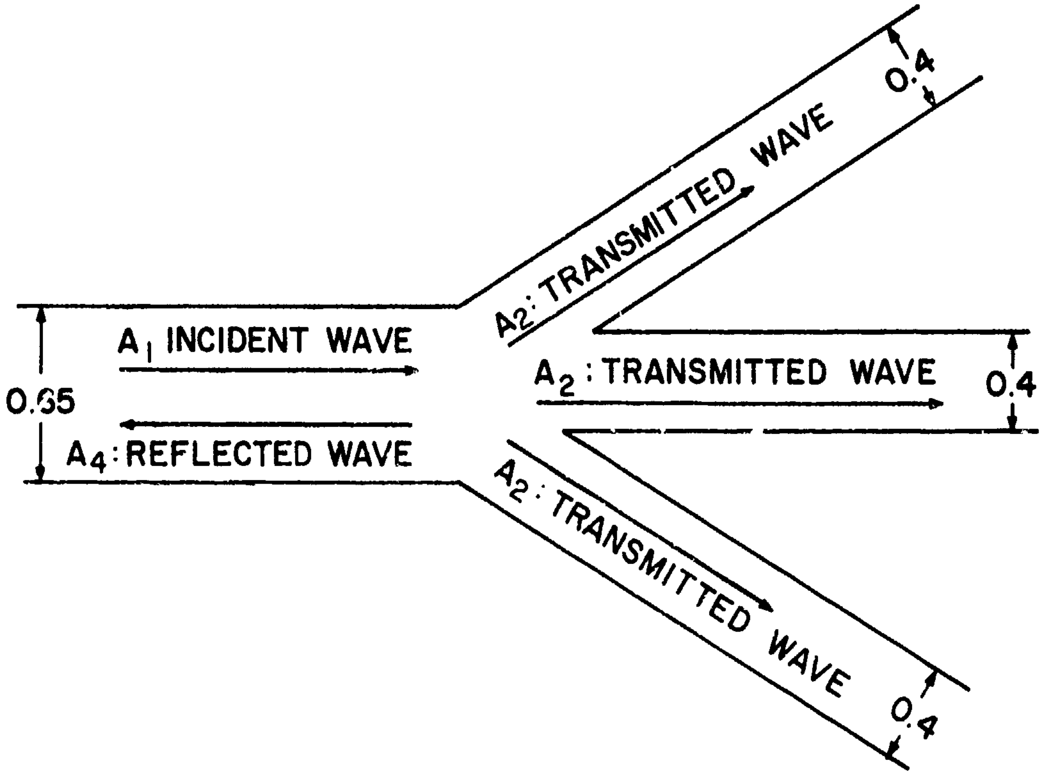

Figure 50. Schematic representing the iliac junction with diameters in cm and the amplitudes A_1, A_2 and A_4 of the incident, transmitted and reflected waves respectively.

STANDING WAVES IN ARTERIAL SYSTEM

Consider a junction or discontinuity where we assume that the following conditions prevail:

1. The reflection coefficient has an amplitude of 10%, $\left|\frac{A_4}{A_1}\right| = 0.1$.
2. There is no phase lag between the incident and reflected pressure waves.
3. The wave velocity is independent of the frequency, α, or the size of the tube.

For incorporating these assumptions, we let the incident wave form traveling in the positive z-direction be described by

$$Ae^{in(t - z/c)}$$

The corresponding reflected wave form traveling in the opposite direction is given

$$\left(\frac{1}{10}\right)Ae^{in(t + z/c)}$$

Thus the resultant of the incident and reflected waves is described by

$$Ae^{in(t - z/c)} + \left(\frac{1}{10}\right)Ae^{in(t + z/c)}$$

This expression may be written as

$$\left(\frac{9}{10}\right)Ae^{in(t - z/c)} + \left(\frac{1}{10}\right)\left[Ae^{in(t - z/c)} + Ae^{in(t + z/c)}\right]$$

(7-24)

According to this expression, we may say that the transmitted wave is made up of the following two components:

1) a component which is 9/10 of the incident wave and
2) another component which is 1/10 of the resultant of the interaction between the incident wave and the reflected wave.

Now we may write the expression (7-24) as

$$\frac{9}{10}Ae^{in(t - z/c)} + \frac{1}{10}Ae^{int}\left[e^{-inz/c} + e^{inz/c}\right]$$

or

$$\frac{9}{10}Ae^{in(t - z/c)} + \frac{1}{10}Ae^{int}\left[\cos\frac{nz}{c} - i\sin\frac{nz}{c} + \cos\frac{nz}{c} + i\sin\frac{nz}{c}\right]$$

or

or $$\frac{9}{10} A e^{in(t - z/c)} + \frac{1}{5} A e^{int} \cos \frac{nz}{c}$$

(7-25)

The right-hand side of equation 7-25 represents the resultant of a wave nine-tenths of the amplitude of the incident wave and a standing wave of one-fifth of its amplitude.

In wave-guide technology this latter method of describing conditions is well known and a method of indicating the efficiency of energy transfer is the standing wave ratio. Since voltage is measured, it is called the voltage standing wave ratio (VSWR). The corresponding quantity in oscillatory fluid flow is the pressure standing wave ratio (PSWR).

The PSWR is measured as follows.

1) If there is no reflected wave, then measurements of the amplitude of pressure variation would indicate the same amplitude at all distances from the junction of tubes.

2) If there is a reflected wave present, then the earlier expression for the resultant wave,

$$\left(\frac{9}{10}\right) Ae^{in(t - z/c)} + \left(\frac{1}{5}\right) Ae^{int} \cos \frac{nz}{c}$$

indicates that the maxima and minima of the amplitude of the oscillatory fluid pressure are located half a wavelength apart.

3) In the extreme case of total reflection, the maxima and minima points of the fluid pressure become nodes and antinodes. The standing wave ratio which is defined as

$$\frac{\text{maximum amplitude of wave}}{\text{minimum amplitude of wave}}$$

is a measure of the efficiency of energy transfer through the junction. In the earlier expression for the resultant wave

$$\left(\frac{9}{10}\right) Ae^{in(t - z/c)} + \left(\frac{1}{5}\right) Ae^{int} \cos \frac{nz}{c}$$

the efficiency of energy transfer through the junction is measured as follows. The maximum amplitude of the transmitted wave is $(9/10)A + (1/5)A = (11/10)A$. The minimum amplitude of the transmitted wave is $(9/10)A$. Thus the pressure standing wave ratio is

$$\frac{\text{maximum amplitude of wave}}{\text{minimum amplitude of wave}} = \frac{(11/10)A}{(9/10)A} = \frac{11}{9} = 1.22$$

If there is total reflection at the junction, with no transmission of energy, the PSWR is infinite.

The method of measurement of the PSWR described above is useless in the arterial system, since it would be impossible to find anywhere a length of artery in which there would be a distance of half a wavelength free from other junctions. There is another approach to the measurement of the PSWR which can be used in the arteries if simultaneous measurements of pressure and pressure gradient are available. This method is described below.

Let the Fourier series for the pressure and pressure gradient be

$$p = p_0 + \sum_m \left(C_m \cos mnt + D_m \sin mnt \right) \tag{7-26}$$

$$= p_0 + \sum_m P_m \cos\left(mnt - \psi_m \right) \tag{7-27}$$

$$-\frac{\partial p}{\partial z} = A_0 + \sum_m \left(A_m \cos mnt + B_m \sin mnt \right) \tag{7-28}$$

$$= A_0 + \sum_m M_m \cos\left(mnt - \varphi_m \right) \tag{7-29}$$

Corresponding to $c_0/c = X - iY$, we write

$$\frac{c_0}{c_m} = X_m - iY_m \tag{7-30}$$

When there is no reflected wave present and the pressure gradient is related to the time rate of change of pressure according to

$$-\frac{\partial p}{\partial z} = \frac{1}{c_m}\frac{\partial p}{\partial t} \tag{7-31}$$

we have, using equation 7-29 for the left-hand side and equation 7-27 for the right-hand side, together with equation 7-30,

$$M_m \cos\left(mnt - \varphi_m\right) = -\frac{mn}{c_m} P_m \sin\left(mnt - \psi_m\right)$$

$$= -\frac{mn}{c_0}\left(\frac{c_0}{c_m}\right) P_m \sin\left(mnt - \psi_m\right)$$

$$= \frac{mn}{c_0}\left(X_m - iY_m\right) P_m \cos\left(mnt - \psi_m + \frac{\pi}{2}\right)$$

It follows that

$$\left| X_m - iY_m \right| = \left(\frac{c_0}{mn}\right) \frac{M_m}{P_m} \qquad (7\text{-}32)$$

and phase $\left\{ X_m - iY_m \right\} = \psi_m - \varphi_m - \frac{\pi}{2}$ (7-33)

Again, using equations 7-26 and 7-28 with equation 7-31, equating real and imaginary parts, we obtain

$$A_m = \frac{mn}{c_0}\left(D_m X_m + C_m Y_m \right) \qquad (7\text{-}34)$$

$$B_m = \frac{mn}{c_0}\left(C_m X_m - D_m Y_m \right) \qquad (7\text{-}35)$$

The relationship between pressure and pressure gradient in the case of no reflected wave is obtained as follows. From the representation of pressure for no reflected wave

$$p = A_1 e^{in(t - z/c)}$$

we have

$$\frac{\partial p}{\partial z} = -\frac{in}{c} A_1 e^{in(t - z/c)}$$

$$-\frac{\partial p}{\partial z} = \left(\frac{in}{c}\right) p$$

If there is a reflected wave present, then the representation for the pressure is

$$p = A_1 e^{in(t - z/c)} + A_2 e^{in(t + z/c)}$$

and the pressure gradient is of the form

$$-\frac{\partial p}{\partial z} = \frac{in}{c}\left[A_1 e^{in(t - z/c)} - A_2 e^{in(t + z/c)} \right]$$

In the presence of a reflected wave, the total wave at any point of the longitudinal axis to the left of the point of reflection is composed of the sum of the incident and the reflected waves. In terms of Fourier series, the total wave is the sum of the Fourier decomposition of the original incident wave and the Fourier decomposition of the reflected wave. Earlier, we had written the pressure for the total wave in the form

$$p = A_1 e^{in(t - z/c)} + A_2 e^{in(t + z/c)}$$

where A_1 and A_2 are complex.

For the m^{th} harmonic of the total pressure wave, we may write

$$p_m = A_{1m} e^{imn(t - z/c)} + A_{2m} e^{imn(t + z/c)}$$

Since the coefficients A_{1m} and A_{2m} are complex, we may write them in the form

$$A_{1m} = P_m' \; e^{-i\psi_m'} \qquad \text{for the incident wave}$$

$$A_{2m} = P_m'' \; e^{-i\psi_m''} \qquad \text{for the reflected wave}$$

Here P_m' and ψ_m' are the modulus and phase of the amplitude of the incident wave and P_m'' and ψ_m'' for the amplitude of the reflected wave. Thus the m^{th} harmonic of the total pressure wave is described by the expression

$$P_m' e^{-i\psi_m'} e^{imn(t - z/c)} + P_m'' e^{-i\psi_m''} e^{imn(t + z/c)} \tag{7-36}$$

As earlier, we take the origin, z=0, at the point of reflection (junction) and back off from the origin a distance, ℓ, $z=\ell$, where the total wave is present and make our measurements. Note that, at the instant of measuring the pressure, time is frozen, t=0, and the place of measurement is at $z=-\ell$. Substituting t=0 and $z=-\ell$ into the expression (7-36), we have

$$P_m' e^{-i\psi_m'} e^{imn\,\ell/c} + P_m'' e^{-i\psi_m''} e^{-imn\,\ell/c}$$

We may denote this expression at the point of measurement by $P_m e^{-i\psi_m}$ and write

or

$$P'_m e^{-i\psi'_m + imn\ell/c} + P''_m e^{-i\psi''_m - imn\ell/c}$$

We may denote this expression in modulus and phase form by $P_m\ e^{-i\psi_m}$ and write

$$P_m e^{-i\psi_m} = P'_m e^{-i\psi'_m + imn\ell/c} + P''_m e^{-i\psi''_m - imn\ell/c} \tag{7-37}$$

Similarly, for the pressure gradient, we have from the relation

$$-\frac{\partial p}{\partial z} = \frac{in}{c}\left[A_1 e^{in(t - z/c)} - A_2 e^{in(t + z/c)}\right]$$

for the mth harmonic

$$-\frac{\partial p_m}{\partial z} = \frac{in}{c_m}\left[P'_m e^{-i\psi'_m} e^{imn(t - z/c)} - P''_m e^{-i\psi''_m} e^{imn(t + z/c)}\right]$$

$$= \frac{in}{c_m}\left[P'_m e^{-i\psi'_m + imn\ell/c} - P''_m e^{-i\psi''_m - imn\ell/c}\right]$$

which is obtained by substituting t=0 and $z=-\ell$ at the point and instant of measurement. We may denote this representation of the pressure gradient in modulus and phase form by $M_m\, e^{-i\phi_m}$ and write

$$M_m e^{-i\varphi_m} = \frac{in}{c_m}\left[P_m' e^{-i\psi_m' + imn\ell/c} - P_m'' e^{-i\psi_m'' - imn\ell/c}\right]$$

(7-38)

In the presence of reflection, the ratio $(M_m/P_m)(c_o/mn)$ describing the amplitude of the m^{th} harmonic of the pressure wave, equation 7-32, now depends on the reflection coefficient at the junction and the distance of the point of measurement from it.

Now we shall obtain the reflection coefficient at the junction when a reflected wave is present. To this end, we divide equation 7-38 by equation 7-37 and obtain

$$\left(\frac{1}{i}\right)\frac{M_m c_m}{P_m n} e^{i(\psi_m - \varphi_m)}$$

$$= \frac{P_m' e^{-i\psi_m' + in\ell/c_m} - P_m'' e^{-i\psi_m'' - in\ell/c_m}}{P_m' e^{-i\psi_m' + in\ell/c_m} + P_m'' e^{-i\psi_m'' - in\ell/c_m}}$$

(7-39)

If we set

$$\left(\frac{M_m}{P_m}\right)\left(\frac{c_m}{in}\right) e^{-i(\psi_m - \varphi_m)} = K \qquad (7\text{-}40)$$

then from equation 7-39 we note that

$$K = \frac{1 - \dfrac{P_m''}{P_m'} \cdot \dfrac{e^{i\psi_m}}{e^{-i\psi_m}} \cdot e^{-2in\ell/c_m}}{1 + \dfrac{P_m''}{P_m'} \cdot \dfrac{e^{-i\psi_m''}}{e^{-i\psi_m'}} \cdot e^{-2in\ell/c_m}} \qquad (7\text{-}41)$$

where the numerator and denominator of the right-hand side of equation 7-39 has been divided by $P_m' \; e^{-i\psi_m' + i\ell/c_m}$. Equation 7-41 may be written as

$$K = \frac{1 - \Delta}{1 + \Delta} \qquad (7\text{-}42)$$

where $$\Delta = \frac{P_m'' e^{-i\psi_m''}}{P_m' e^{-i\psi_m'}} \cdot e^{-2in\ell/c_m}$$

From equation 7-42 it follows that

$$\Delta = \frac{1 - K}{1 + K}$$

$$\text{or} \quad \left(\frac{P_m'' e^{-i\psi_m''}}{P_m' e^{-i\psi_m'}} \right) e^{-2in\ell/c_m} = \frac{1-K}{1+K} \qquad (7\text{-}43)$$

The quantity $\frac{P_m'' \, e^{-i\psi_m''}}{P' \, e^{-i\psi_m'}}$ is the reflection coefficient of the junction in complex form, i.e., it is the same as the quantity A_4/A_1 of equation 7-15

$$\frac{A_4}{A_1} = \frac{1-\lambda}{1+\lambda} \qquad (7\text{-}15)$$

Combining equations 7-15 and 7-43 we have

$$\frac{1-\lambda}{1+\lambda} = \left(\frac{1-K}{1+K}\right) e^{2in\ell/c_m} \qquad (7\text{-}44)$$

If, therefore, the geometry of the junction is known, the theory can be tested by calculating λ and K and attempting to find a c_o which is consistent for all harmonics.

DISCONTINUITY DUE TO ELECTROMAGNETIC FLOWMETER I

There is another type of simple discontinuity which has the opposite effect from that of a junction or a constriction, and which has an important practical application. Some types of electromagnetic flowmeters require the insertion in the artery of a short length of rigid tube, or may confine the artery by means of a cuff. The effect of such an artifact on steady flow is negligible. However, if the flow has large oscillatory components, distortion is introduced.

Consider a tube, elastic for the most part, which has in it a stationary, rigid portion of length ℓ. See figure 51. Suppose

1) the incident pressure wave is described by $A_1\, e^{in(t-z/c)}$ in the elastic portion;
2) the reflected pressure wave is described by $A_4\, e^{in(t-z/c)}$ in the elastic portion;
3) at the incident end of the rigid portion, $z = 0$;
4) at $z=0$ the pressure wave is described by $A_2\, e^{int}$, which is obtained from $A_2\, e^{in(t-z/c)}$ by setting $z=0$;
5) at $z=\ell$ the pressure wave is described by $A_3\, e^{int}$. This is obtained from $A_3\, e^{in(t-z/c)}$ by setting $z=\ell$ and considering ℓ/c negligible, since c in the rigid portion is numerically much larger than ℓ. This is correct, since the transmission velocity in the rigid portion is infinite.

For continuity of pressure across the discontinuity,

1) the transmitted pressure wave is described by $A_3\, e^{in(t-z/c)}$ and
2) $A_1 + A_4 = A_2$ (7-45)

For continuity of flow across the discontinuity, we must have for the elastic-rigid-elastic tube

$$\left(\frac{A_1 - A_4}{\rho c}\right) M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)} = \left(\frac{A_2 - A_3}{in\rho\ell}\right) M'_{10}(\alpha)\, e^{i\varepsilon'_{10}(\alpha)} = \left(\frac{A_3}{\rho c}\right) M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)} \tag{7-46}$$

In equation 7-46, note that the pressure gradient in the rigid portion is $(A_2 - A_3)/\ell$. In the elastic-rigid portion, the change from $1/\rho c$ to $1/in\rho\ell$, is on account of the fact that in the elastic portion the flow is due to pressure and in the rigid portion the flow is due to pressure gradient. Furthermore, since the artery is assumed to be of the same diameter throughout (across the discontinuity), the value of $\alpha(= R\sqrt{\frac{n}{\nu}})$ is the same at all points of the artery.

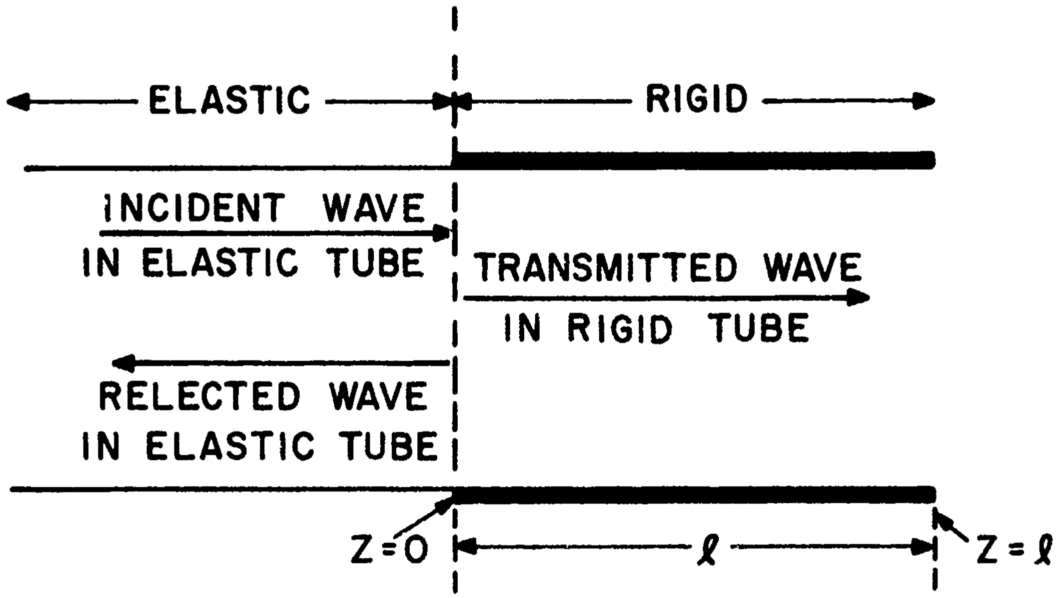

Figure 51. Discontinuity due to either the insertion of a short length of rigid tube into the artery or the confinement of the artery by means of a cuff.

From equation 7-46 we write

$$A_1 - A_4 = \frac{c}{in\ell}\left(A_2 - A_3\right)\frac{M'_{10}(\alpha)\, e^{i\varepsilon'_{10}(\alpha)}}{M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)}} \tag{7-47}$$

Now $A_1 + A_4 = A_2$ and $A_1 - A_4 = A_3$. Subtracting, we obtain

$$2A_4 = A_2 - A_3$$

Using this relation in equation 7-47 we have

$$A_1 - A_4 = \left(\frac{2c}{in\ell}\right)A_4\,\frac{M'_{10}(\alpha)\, e^{i\varepsilon'_{10}(\alpha)}}{M''_{10}(\alpha)\, e^{i\varepsilon''_{10}(\alpha)}} \tag{7-48}$$

Dividing both sides of equation 7-48 by A_4, we obtain

$$\frac{A_1}{A_4} = 1 + \left(\frac{2c}{in\ell}\right) \frac{M_{10}'(\alpha)}{M_{10}''(\alpha)} \cdot e^{i\left[\varepsilon_{10}'(\alpha) - \varepsilon_{10}''(\alpha)\right]} \tag{7-49}$$

Since $\frac{A_3}{A_1} = \frac{A_1 - A_4}{A_1} = 1 - \frac{A_4}{A_1}$

we have from equation 7-49

$$1 - \frac{A_4}{A_1} = 1 - \frac{1}{1 + \left(\frac{2c}{in\ell}\right) \frac{M_{10}'}{M_{10}''} e^{i(\varepsilon_{10}' - \varepsilon_{10}'')}} \tag{7-50}$$

For the limiting condition of stiff constraint, $M_{10}'/M_{10}'' = 1$ and $\varepsilon_{10}' = \varepsilon_{10}''$, and equation 7-50, describing the ratio of the transmitted wave to the incident wave, simplifies to the form

$$\frac{A_3}{A_1} = 1 - \frac{A_4}{A_1} = \frac{1}{1 + \frac{in\ell}{2c}}$$

or

$$\frac{A_3}{A_1} = \frac{1}{1 + \frac{in\ell}{2c_o}\left(\frac{c_o}{c}\right)} \tag{7-51}$$

The pressure gradient in the rigid portion, $(A_2 - A_3)/\ell$, may also be found in terms of the magnitude of the incident pressure wave, A_1, as follows. From equation 7-46, for the limiting condition of stiff constraint, we write

$$\frac{A_1 - A_4}{\rho c} = \frac{A_2 - A_3}{in\rho \ell}$$

Dividing both sides by A_1, we obtain

$$\frac{A_2 - A_3}{A_1 \, in\rho \ell} = \left(\frac{A_1 - A_4}{A_1}\right)\frac{1}{\rho c} = \left(1 - \frac{A_4}{A_1}\right)\frac{1}{\rho c}$$

Combining this result with equation 7-51, we have

$$\frac{A_2 - A_3}{in\rho \ell} = \left(\frac{A_1}{\rho c}\right)\frac{1}{1 + \frac{in\ell}{2C_o}\left(\frac{c_o}{c}\right)} \tag{7-52}$$

In equation 7-52, note that the left-hand side describes the flow in the rigid portion (flowmeter portion), the factor $A_1/\rho c$ describes the incident

flow and $\left| \dfrac{1}{1 + \frac{in\ell}{2C_o}\left(\frac{c_o}{c}\right)} \right| \leqq 1$

Therefore the rate of flow as measured by the flowmeter is reduced, i.e.,

$$\frac{A_2 - A_3}{in\rho \ell} \leqq \frac{A_1}{\rho c}$$

Thus, from the flowmeter observation, we should be able to calculate what the flow would have been, had it not been distorted by the flowmeter.

As an example, suppose the artery is confined in a cuff which is 15 mm long. In the femoral artery of the dog, taking the wave velocity, $c = 450$ cm/sec, and a pulse frequency of 3 cycles per second, i.e., $n = 2\pi f = (2\pi)3 = 6\pi$, we find that the ratio of the amplitude of the transmitted wave to the incident wave is

$$\frac{A_3}{A_1} = \frac{1}{1 + \frac{in\ell}{2c}} = \frac{1}{1 + \frac{i(6\pi)(1.5)}{2(450)}} = \frac{1}{1 + \frac{i\pi}{100}} \tag{7-53}$$

Since this result is good for all the harmonics, we may write

$$\left(\frac{A_3}{A_1}\right)_m = \frac{1}{1 + \frac{inm\ell}{2c}} \quad \text{and get} \quad \left(\frac{A_3}{A_1}\right)_m = \frac{1}{1 + \frac{i\pi m}{100}}$$

where m is the order of the harmonic. We have neglected the ratio c_0/c in equation 7-53, since the calculation is only intended to show the order of magnitude of the effect of introducing the cuff. For the *4th* harmonic, the reduction in amplitude is about 1% and the phase lag about 7°. Thus for this instrument the effect of the cuff is negligible.

In practice, electromagnetic flowmeters have been used with flexible plastic tubes leading from a severed artery in which the effective length of the rigid insert is 15 cm or more. Table VI indicates the magnitude of the ratio A_3/A_1 and the phase lag for the first four harmonics in the femoral artery of the dog. From the results shown in table VI we conclude that an electromagnetic flowmeter with a rigid insert of 15 cm or more cannot represent normal conditions in the artery. Any rigid insert or cuff which confines

TABLE VI

The Amplitude and Phase Lag of the Ratio of the Transmitted Wave to the Incident Wave for the First Four Harmonics of the Pulse Wave in the Femoral Artery of the Dog

Harmonic m	$\left\lvert\frac{A_3}{A_1}\right\rvert$	$-$ Phase $\{\frac{A_3}{A_1}\}$
1	0.946	19°
2	0.847	32.15°
3	0.728	43.3°
4	0.643	51.5°

the artery, acts as a low-pass filter and introduces both phase and amplitude distortion of the pulse wave. Therefore, it is of the greatest importance that inserts of this kind should be kept as short as possible.

We will now show details of calculations that have to be made for corrections of observations of flow with an electromagnetic flowmeter having a rigid insert.

Suppose that the observed flow, F(t), made with an electromagnetic flowmeter can be represented by the Fourier series

$$F(t) = A_o + \sum_m (A_m \cos mnt + B_m \sin mnt) \tag{7-54}$$

For the corrected flow, G(t), we write to a first approximation

$$G(t) = F(t) + \frac{dF}{dt} (\Delta t) \tag{7-55}$$

where $\Delta t = \ell/c$ = amount of time it takes for the flow to travel over the distance of measurement downstream from the origin, z=0.

Suppose that the corrected flow can be represented by

$$G(t) = A_o + \sum_m (A'_m \cos mnt + B'_m \sin mnt) \tag{7-56}$$

From equations 7-54 and 7-55 we write

$$G(t) = F(t) + \frac{dF}{dt}\left(\frac{\ell}{c}\right)$$

$$= A_o + \sum_m (A_m \cos mnt + B_m \sin mnt)$$

$$+ \left(\frac{\ell}{c}\right)\sum_m \left| - A_m(mn) \sin mnt + B_m(mn) \cos mnt \right| \qquad (7\text{-}57)$$

Equating the coefficients of cos (mnt) and sin (mnt) appearing on the right-hand sides of equations 7-56 and 7-57, we find that

$$A_m{}' = A_m + (2\pi f)m\left(\frac{\ell}{c}\right)B_m \qquad (7\text{-}58)$$

$$B_m{}' = B_m - (2\pi f)m\left(\frac{\ell}{c}\right)A_m \qquad (7\text{-}59)$$

Thus, in the Fourier series for the corrected flow, equation 7-56, we use the values of the corrected coefficients $A_m{}'$ and $B_m{}'$ as given by equations 7-58 and 7-59.

DISCONTINUITY DUE TO ELECTROMAGNETIC FLOWMETER II

In another type of electromagnetic flowmeter, the artery is pressed between the poles of the magnet. The diameter of the artery across the gap is reduced by about 20% but left free to expand in the perpendicular direction. It is claimed that this constriction in diameter will have a trivial effect on the rate of flow, since the cross-sectional area is reduced by only 6%. Although the present theory does not take into account any effect on the flow produced by the change in shape due to the lateral compression of the artery, we shall calculate the reflection produced by this order of change in area due to the artery being pressed between the poles of a magnet.

Accordingly, we make the following assumptions:

(1) Let the width of the pole pieces of the magnet be ℓ cm. The pressure gradient is measured over the tube length ℓ.

(2) The velocity of the pressure wave is unchanged by the lateral compression of the artery.

(3) The width ℓ is so small that repeated reflections at the two ends will be taken into account.

(4) The fluid pressure in the absence of the constriction (due to flowmeter) is given by $A_1 e^{in(t - z/c)}$.

(5) The origin of the longitudinal axis, z=0, is at the point of the constriction, i.e., at the proximal end of the narrower portion of the tube.

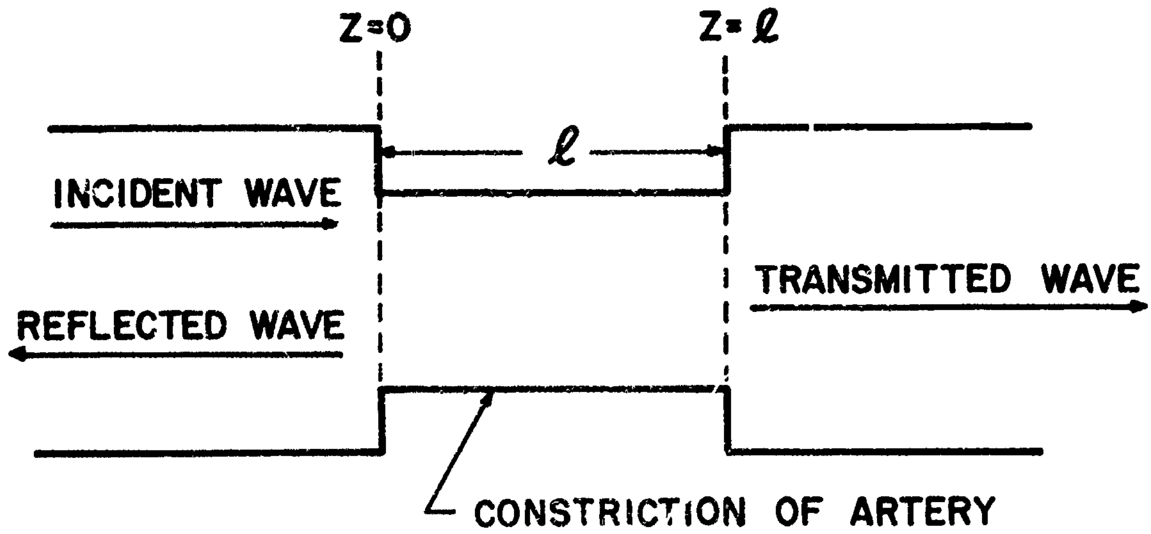

Figure 52. Schematic representing the constriction of the artery by the poles of a magnet.

We recall equation 7-10 for the reflection coefficient λ

$$\frac{A_1 - A_4}{A_2} = \lambda = \left(\frac{r}{R}\right)^{2.5}\left[\frac{M'_{10}(\alpha_2)}{M'_{10}(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon'_{10}(\alpha_2) - \varepsilon'_{10}(\alpha_1)\right]} \tag{7-10}$$

For a 6% reduction in cross-sectional area due to the constriction, equation 7-10 has the form

$$\lambda_{\text{PROXIMAL}} = (0.94)\left[\frac{M'_{10}(\alpha_2)}{M'_{10}(\alpha_1)}\right]^{1/2} e^{\frac{i}{2}\left[\varepsilon'_{10}(\alpha_2) - \varepsilon'_{10}(\alpha_1)\right]} \tag{7-60}$$

Equation 7-60 gives the value of λ at the proximal end of the narrower portion of the tube. At the distal end of the constriction, because of the inversion of the cross-sectional area, the value of λ will be

$$\lambda_{\text{distal}} = \frac{1}{\lambda_{\text{proximal}}}$$

If A_1 is the amplitude of the incident pressure wave and A_4 is the amplitude of the reflected wave at the proximal end of the constriction then, as earlier, we have

$$\frac{A_4}{A_1} = \frac{1 - \lambda}{1 + \lambda} \qquad (7\text{-}15)$$

If A_3 is the amplitude of the transmitted pressure wave (transmitted through the constriction), then from the continuity of pressure across the constriction, we write

$$A_1 + A_4 = A_3$$

Dividing through by A_1, we have

$$1 + \frac{A_4}{A_1} = \frac{A_3}{A_1}$$

or

$$1 + \frac{1 - \lambda}{1 + \lambda} = \frac{A_3}{A_1}$$

or

$$\frac{A_3}{A_1} = \frac{2}{1 + \lambda}$$

When this transmitted pressure wave reaches the distal end of the constriction, $z=\ell$, its amplitude will be modified according to

$$A_3 = A_1\left(\frac{2}{1+\lambda}\right) e^{-int} \qquad (7\text{-}61)$$

Since $t = \dfrac{\text{length of constriction traveled by wave}}{\text{velocity of wave}} = \dfrac{\ell}{c}$, we write equation 7-61 in the form

$$A_3 = A_1\left(\frac{2}{1+\lambda}\right) e^{-\frac{in\ell}{c}}$$

At the distal end of the constriction, this transmitted wave will give rise to a reflected wave of amplitude A_3 given by

$$A_3 = A_1\left(\frac{2}{1+\lambda}\right)e^{-\frac{in\ell}{c}}\left[-\left(\frac{1-\lambda}{1+\lambda}\right)\right] = A_1\left(\frac{\lambda-1}{1+\lambda}\right)\left(\frac{2}{1+\lambda}\right)e^{-\frac{in\ell}{c}}$$

Here A_3 represents the reflected wave from the distal end and A_1 is the original incident wave. In traveling back again to the proximal end, this reflected wave will be attenuated and described by

$$\left[A_1\left(\frac{\lambda-1}{1+\lambda}\right)\left(\frac{2}{1+\lambda}\right)e^{-\frac{in\ell}{c}}\right]e^{-\frac{in\ell}{c}}$$

or

$$A_1\left(\frac{\lambda-1}{1+\lambda}\right)\left(\frac{2}{1+\lambda}\right)e^{-\frac{2in\ell}{c}}$$

At the proximal end of the constriction, this wave will be transmitted back into the larger tube with amplitude

$$A_1\left(\frac{\lambda-1}{1+\lambda}\right)\left(\frac{2}{1+\lambda}\right)\left(\frac{2\lambda}{1+\lambda}\right)e^{-\frac{2in\ell}{c}} \qquad \text{(a)}$$

The factor $2\lambda/(1+\lambda)$ appearing above may be explained as follows. Initially, both the incident wave and the transmitted wave were going in the same direction and we used the continuity condition

$$A_1 + A_4 = A_3$$

to obtain

$$\frac{A_3}{A_1} = 1 + \frac{A_4}{A_1} = \frac{2}{1 + \lambda}$$

Now, on account of reflection, we have to use the continuity condition

$$-A_1 + A_4 = -A_3$$

from which we obtain

$$\frac{A_3}{A_1} = 1 - \frac{A_4}{A_1} = 1 - (\frac{1 - \lambda}{1 + \lambda}) = \frac{2\lambda}{1 + \lambda}$$

For higher orders of reflection, i.e., when the same wave has gone through several reflections, each time the wave traverses the constriction in both directions, it reappears at the proximal end as a reflected wave with its amplitude reduced in the ratio

$$\frac{2\lambda(\lambda-1)}{(1+\lambda)^2} \cdot e^{-2in\ell/c} \qquad \text{(b)}$$

Comparing the wave forms given by (a) and (b) above, note that the amplitude of the transmitted wave at z=0, going to the right is

$$A_1 (\frac{2}{1 + \lambda})$$

and the amplitude of the reflected wave transmitted back (returning back) into the large tube is described by form (a).

Since these waves are going on and on, back and forth, it follows that if we add together all the reflected waves except the first (using this as a reference term) we obtain a geometric progression whose first term is

$$A_1 \left(\frac{2}{1+\lambda}\right)\left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{\lambda+1}\right) e^{-2in\ell/c}$$

and whose common ratio is

$$\frac{2\lambda(\lambda-1)}{(1+\lambda)^2} \cdot e^{-2in\ell/c}$$

We recall that the first reflected wave at the proximal end of the constriction has an amplitude of

$$A_1\left(\frac{2}{1+\lambda}\right)\left(\frac{\lambda-1}{\lambda+1}\right) e^{-2in\ell/c}$$

As mentioned above, we neglect this wave in the geometric progression and use it as the reference term. The first reflected wave at the proximal end of constriction will be transmitted back into the larger tube with amplitude

$$A_1\left(\frac{2}{1+\lambda}\right)\left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{\lambda+1}\right) e^{-2in\ell/c}$$

This is the first term of the geometric progression. Again, when the last wave above is reflected, its amplitude becomes

$$\left[A_1\left(\frac{2}{1+\lambda}\right)\left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{1+\lambda}\right) e^{-2in\ell/c}\right]\left[\left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{1+\lambda}\right) e^{-2in\ell/c}\right]$$

= (first term of geometric progression) (common ratio)

The sum of the geometric series given by $\frac{\text{first term}}{1 - \text{common ratio}}$, becomes

$$\frac{A_1\left(\frac{2}{1+\lambda}\right)\left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{\lambda+1}\right) e^{-2in\ell/c}}{1 - \left(\frac{\lambda-1}{\lambda+1}\right)\left(\frac{2\lambda}{1+\lambda}\right) e^{-2in\ell/c}}$$

or $$A_1\left(\frac{2}{1+\lambda}\right)\frac{2\lambda(\lambda-1)e^{-2in\ell/c}}{(\lambda+1)^2 - 2\lambda(\lambda-1)e^{-2in\ell/c}}$$

We recall that the equation for continuity of flow in the constrained tube is given by equation 7-7. Equation 7-7, under the assumption that the mass loading is the same for the two tubes, has the form given by equation 7-10. Now, in calculating the flow to the right of the constriction, the sum of all the amplitudes of the reflected terms must be subtracted from A_1, the amplitude of the incident pressure wave, i.e., A_1 - (A_4 + sum of all the other A_4s). Here A_1 is the amplitude of the incident wave going to the right, A_4 is the amplitude of the first reflected wave to the left which was used as a reference term and left out. The sum of all the other A_4s, i.e., the sum of all the other reflected waves is given by the sum of the geometric progression. The above statement may be written as

$$A_1 - \left(\frac{1-\lambda}{1+\lambda}\right)A_1 - A_1\left(\frac{2}{1+\lambda}\right)\left[\frac{2\lambda(\lambda-1)e^{-2in\ell/c}}{(1+\lambda)^2 - 2\lambda(\lambda-1)e^{-2in\ell/c}}\right]$$

For the ratio

$$\frac{\text{flow in tube with the constriction}}{\text{flow in tube without the constriction}}$$

we write

$$\frac{A_1 - \left(\frac{1-\lambda}{1+\lambda}\right)A_1 - A_1\left(\frac{2}{1+\lambda}\right)\left[\frac{2\lambda(\lambda-1)\,e^{-2in\ell/c}}{(1+\lambda)^2 - 2\lambda(\lambda-1)\,e^{-2in\ell/c}}\right]}{A_1}$$

$$= 1 - \left(\frac{1-\lambda}{1+\lambda}\right) - \left(\frac{2}{1+\lambda}\right)\left[\frac{2\lambda(\lambda-1)\,e^{-2in\ell/c}}{(1+\lambda)^2 - 2\lambda(\lambda-1)\,e^{-2in\ell/c}}\right]$$

$$= \frac{2\lambda}{1+\lambda} - \left(\frac{2}{1+\lambda}\right)\left[\frac{2\lambda(\lambda-1)\,e^{-2in\ell/c}}{(1+\lambda)^2 - 2\lambda(\lambda-1)\,e^{-2in\ell/c}}\right]$$

$$= \frac{2\lambda}{1+\lambda}\left\{1 - \left[\frac{2(\lambda-1)\,e^{-2in\ell/c}}{(1+\lambda)^2 - 2\lambda(\lambda-1)\,e^{-2in\ell/c}}\right]\right\} \tag{7-62}$$

We now assume that the flowmeter constricts the tube by only a small fraction of its original total area. Accordingly we write

$$\lambda = 1 - \delta$$

where δ is small and neglect second and higher powers of δ appearing in the terms $(1+\lambda)^2$, $2\lambda(\lambda-1)$ of equation 7-62. Thus

$$2\lambda = 2(1 - \delta)$$

$$1 + \lambda = 1 + (1 - \delta) = 2 - \delta$$

$$2(\lambda - 1) = 2(1 - \delta - 1) = -2\delta$$

$$(1 + \lambda)^2 = (1 + 1 - \delta)^2 = (2 - \delta)^2 \simeq 4 - 4\delta$$

$$2\lambda(\lambda - 1) = 2(1 - \delta)(1 - \delta - 1) = -2\delta + 2\delta^2 \simeq -2\delta$$

and the flow ratio described by equation 7-62 has the form

$$\text{flow ratio} = \left\{\frac{2(1-\delta)}{2-\delta}\right\}\left\{1 - \left[\frac{(-2\delta)e^{-2in\ell/c}}{(4-4\delta) + 2\delta e^{-2in\ell/c}}\right]\right\}$$

$$= \frac{2(1-\delta)}{2(1-\delta/2)}\left\{1 + \frac{2\delta e^{-2in\ell/c}}{4(1-\delta) + 2\delta e^{-2in\ell/c}}\right\}$$

$$= \left(\frac{1-\delta}{1-\delta/2}\right)\left\{1 + \frac{\delta e^{-2in\ell/c}}{2(1-\delta) + \delta e^{-2in\ell/c}}\right\} \tag{7-63}$$

The first factor in equation 7-63 may be written as

$$\frac{1 - \delta}{1 - \frac{\delta}{2}} = 1 - \frac{\delta}{2}$$

up to first order. In the second factor, the maximum value of $e^{-2in\ell/c}$ is 1. Therefore, we may write

$$1 + \frac{\delta e^{-2in\ell/c}}{2(1-\delta) + \delta e^{-2in\ell/c}} \cong 1 + \frac{\delta}{2-\delta} \cong 1$$

since δ is small compared with 2. Thus, equation 7-63 reduces to the form

$$\text{flow ratio} = 1 - \frac{\delta}{2} \tag{7-64}$$

From equation 7-64 we conclude that the effect of the flowmeter on the flow is very small for a slight constriction of short length.

SECTION VIII

CORRECTION FOR THE OSCILLATORY VARIATION IN TUBE DIAMETER

INTRODUCTION

Earlier, from the point of view of motion of the liquid, we had regarded the diameter of the tube as constant. Actually, at any instant, the cross section of the elastic tube must be considered to be deformed. We make this correction for the oscillatory variation in the tube radius in the equations of liquid motion, assume that the lines of laminar flow expand and contract with the artery and neglect inertia terms and second order effects of the longitudinal fluid velocity. We seek a solution of the resulting equation with the pressure gradient, longitudinal velocity and tube deformation expressed in the form of a Fourier series. We also obtain corrections for the interactions between the harmonic components and apply the results to arterial flow.

HARMONIC REPRESENTATIONS FOR $\frac{\partial p}{\partial z}$, w, and $\frac{2\xi}{R}$

Up to this point, the volume rate of flow, Q, and the average longitudinal fluid velocity, $\bar{w}$, have been used freely as indicators of "flow," under the assumption that

$$Q = (\pi R^2)\bar{w} = (\pi R^2)\frac{A_1}{\rho_0 c}(1 + \eta F_{10})e^{int} \tag{8-1}$$

where R denotes the constant tube radius. However, equation 8-1 is approximately true only since at any time the radius of the elastic tube is not R but R+ξ, and ξ varies with time. Thus, at any instant, the cross-sectional area due to a change, ξ, along the radius is

$$(R + \xi)^2 = \pi(R^2 + 2\xi R + \xi^2) = \pi(R^2 + 2\xi R)$$

up to first order in the correction ξ. Moreover, we may write

$$\pi(R^2 + 2\xi R) = \pi R^2 + 2\pi\xi R = \pi R^2[1 + (2\xi/R)]$$

Note that at any instant, the cross section of the tube is made up of the constant cross section πR^2 and a first-order change in the cross section $2\pi\xi R$.

Taking into account the radial change, ξ, we may write a better approximation for the volume rate of flow as

$$Q = \pi R^2\left(1+\frac{2\xi}{R}\right)\bar{w} = \pi R^2\left(1+\frac{2\xi}{R}\right)\frac{A_1}{\rho_0 c}\left(1+\eta F_{10}\right)e^{int} \tag{8-2}$$

Inserting the value $\bar{w} = \frac{2\xi}{R}$ from equation 6-33, we have

$$Q = \pi R^2(1 + \frac{\bar{w}}{c})\bar{w} \tag{8-3}$$

Even this representation for Q, equation 8-3, is not fully corrected for the oscillatory variation in the radius. In what follows, we shall allow for this variation in the radius.

Recall the longitudinal equation of motion of the fluid when the tube radius is considered constant

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z} = -\,\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} + \nu\left|\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right| \tag{3-28}$$

In equation 3-28 we change the independent variable according to $y = r/R$ and replace R by $R+\xi$. Moreover, we delete the inertia terms $u(\partial w/\partial r)$ and $w(\partial w/\partial z)$. Furthermore, the second-order change in w, $\partial^2 w/\partial z^2$, is neglected because it is of order n^2R^2/c^2. Thus equation 3-28 reduces to the form

$$\frac{1}{\nu}\,\frac{\partial w}{\partial t} = -\,\frac{1}{\rho_0 \nu}\,\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} \tag{8-4}$$

Since

$$\frac{1}{r}\,\frac{\partial w}{\partial r} = \frac{1}{R^2 y}\,\frac{\partial w}{\partial y}$$

and

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \frac{1}{R^2}\,\frac{\partial^2 w}{\partial y^2} \quad ,$$

equation 8-4 has the form

$$\frac{1}{\nu}\frac{\partial w}{\partial t} = -\frac{1}{\rho_0 \nu}\frac{\partial p}{\partial z} + \frac{1}{R^2}\frac{\partial^2 w}{\partial y^2} + \frac{1}{R^2 y}\frac{\partial w}{\partial y}$$

or

$$\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{R^2}{\nu}\frac{\partial w}{\partial t} = \frac{R^2}{\mu}\frac{\partial p}{\partial z} \tag{8-5}$$

Making the change from R^2 to $R^2(1 + \frac{2\xi}{R})$ in equation 8-5, we obtain the equation of motion of the fluid corrected for the radial expansion in the form

$$\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{R^2}{\nu}(1 + \frac{2\xi}{R})\frac{\partial w}{\partial t} = \frac{R^2}{\mu}(1 + \frac{2\xi}{R})\frac{\partial p}{\partial z} \tag{8-6}$$

If we seek a solution of equation 8-6 which is of the same form as the solution of equation 3-28 for a constant radius, we can imagine the quantities $\partial p/\partial z$, w, and $2\xi/R$ in equation 8-6 represented in terms of Fourier series in $n(t-z/c)$. As a result, the products of the Fourier series can b multiplied out and a set of equations representing the fluid velocity components obtained by collecting up corresponding terms. However, the wave velocity, c, is not the same at all frequencies, i.e., c is a function of the harmonics m = 1,2,3,... . In other words, each harmonic in the Fourier series has a particular c associated with it, and on multiplying two periodic terms together, there will be some exponential terms "left over," as it were, which would disappear (being equal to unity) in a system with constant wave velocity.

Suppose the Fourier expansions of two functions, f_1 and f_2, are given respectively by

$$f_1 = \sum_m A_{1,m}\, e^{imn(t - z/c_m)}$$

$$f_2 = \sum_p A_{2,p}\, e^{ipn(t - z/c_p)}$$

then the Fourier expansion of the product of f_1 and f_2 is given by

$$F = f_1 f_2 = \sum_{m,p} A_{1,m} A_{2,p} e^{in\left[t(m+p) - z\left(\frac{m}{c_m} + \frac{p}{c_p}\right)\right]} \tag{8-7}$$

$$= \sum_{m,p} A_{1,m} A_{2,p} e^{in\left[t(m+p) - \frac{z}{c_s}(m+p)\right]}$$

$$= \sum_{m,p} A_{1,m} A_{2,p} e^{in\left[(m+p)(t - z/c_s)\right]}$$

if $c_m = c_p$. We call the common value of c_m and c_p by c_s. This representation for the product $f_1 f_2$ may be written in the form

$$f_1 f_2 = \sum_s A_s e^{isn(t - z/c_s)} \tag{8-8}$$

when the wave velocity is the same for different harmonics. We note that the above representation, equation 8-8, has the same form as the expansions used for the functions p, u and w in the solution of equation 3-28.

We compare the two expansions, equation 8-7 for $c_m \neq c_p$ and equation 8-8 for $c_m = c_p$, to see how far off we are in using the regular Fourier expansion.

For example, if the sum of the harmonics appearing in the two functions f_1 and f_2 is $s = m + p = 4$, then the combinations of m and p which add up to 4 may be tabulated as follows:

m	p
0	4
1	3
2	2
3	1
4	0

Note that the two sets of values given by $m = 0$, $p = 4$ and $m = 4$, $p = 0$ satisfy both representations (8-7 and 8-8 above).

Let us denote the dist[illegible]t values of c corresponding to the first four harmonics by c_1, c_2, c_3 and c_4. Comparing the two representations (8-7 and 8-8), i.e., taking the ratios, we find that

$$\frac{e^{in\left[t(m+p) - z\left(m/c_m + p/c_p\right)\right]}}{e^{ins\left[t - z/c_s\right]}}$$

and if $s = m + p = 2$, with $m = 1$ and $p = 1$, we have

$$e^{1/c_2 - 1/c_1}$$

When $s = m + p = 3$, with $m = 1$ and $p = 2$ or with $m = 2$ and $p = 1$, we have

$$e^{3/c_3 - 2/c_2 - 1/c_1}$$

When $s = m + p = 4$, with $m = 2$ and $p = 2$, we have

$$e^{1/c_4 - 1/c_2}$$

When $s = m + p = 4$, with $m = 1$ and $p = 3$ or with $m = 3$ and $p = 1$, we have

$$e^{4/c_4 - 3/c_3 - 1/c_1}$$

Thus the quantities of interest are the following

1) $\dfrac{1}{c_2} - \dfrac{1}{c_1}$

2) $\dfrac{3}{c_3} - \dfrac{2}{c_2} - \dfrac{1}{c_1}$

3) $\dfrac{1}{c_4} - \dfrac{1}{c_2}$

4) $\dfrac{4}{c_4} - \dfrac{3}{c_3} - \dfrac{1}{c_1}$

From figure 24 we note the variation of c_1/c_0 with respect to α and, in particular, for $\alpha > 3$ the variation in the value of c is small. Therefore, it is reasonable to use a simple perturbation technique, with ξ as the perturbation parameter, to solve equation 8-6.

This correction of the linear solution, to provide for the finite expansion in tube diameter, is the simplest correction to be made. Moreover, we obtain a better perspective for the more important correction for the inertia terms. Although, at first sight, the correction due to ξ may appear less important in principle than the inertia-term correction, it may well be equally important in magnitude, for it may cope with fairly large arterial distentions, such as occur near the heart, without the mathematical difficulties

that arise in the analysis of finite strain. Actually, since both corrections are concerned with terms of the order $\bar{w}/c$, they are equally important. In fact, it can be shown that for moderate values of α they are of the same order of magnitude.

We seek a solution of equation 8-6 when the pressure gradient is represented by a Fourier series of four harmonics, together with a constant term which will be assumed to give a Poiseuille flow, the static expansion of the tube (which would give a tapering effect) being neglected. This is justified, since this constant term is small, being less than one-eighth of the largest oscillatory terms. The detailed solution will be developed for two harmonics only, since this illustrates the method adequately without wearisome repetition. Since equation 8-6 is nonlinear, we may no longer write the pressure gradient as

$$\frac{\partial p}{\partial z} = A_0 + A_1\, e^{int} + A_2\, e^{2int} + \ldots \tag{8-9}$$

and take the real part, otherwise half the interaction terms will be lost. It is necessary to start from the pressure gradient in real form and write down its exact complex equivalent. Accordingly, we assume that

$$\frac{\partial p}{\partial z} = M_0 + M_1 \cos(nt + \phi_1) + M_2 \cos(2nt + \phi_2) \tag{8-10}$$

Next, we define $A_0, A_1, A_2, \ldots$ by

$$A_0 = M_0$$

$$A_1 = \frac{1}{2} M_1\, e^{i\phi_1}$$

$$A_2 = \frac{1}{2} M_2\, e^{i\phi_2}$$

Now,

$$M_1 \cos(nt + \varphi_1) = M_1 \left[\cos\varphi_1 \cos nt - \sin\varphi_1 \sin nt\right]$$

$$= M_1 \left[\cos\varphi_1 \left(\frac{e^{int} + e^{-int}}{2}\right) - \sin\varphi_1 \left(\frac{e^{int} - e^{-int}}{2i}\right)\right]$$

$$= \frac{M_1}{2}\left[\left(\cos\varphi_1 + i\sin\varphi_1\right)e^{int} + \left(\cos\varphi_1 - i\sin\varphi_1\right)e^{-int}\right]$$

$$= \frac{M_1}{2} e^{i\varphi_1} e^{int} + \frac{M_1}{2} e^{-i\varphi_1} e^{-int}$$

Note that $e^{-i\phi_1}$ and e^{-int} are complex conjugates of $e^{i\phi_1}$ and e^{int}. If we denote $\frac{M_1}{2} e^{i\phi_1}$ by A_1, then $\frac{M_1}{2} e^{-i\phi_1}$ will be denoted by $A_1{*}$. Thus,

$$M_1 \cos(nt + \phi_1) = A_1 e^{int} + A_1{*} e^{-int}$$

Similarly

$$M_2 \cos(2nt + \phi_2) = A_2 e^{2int} + A_2{*} e^{-2int}$$

Therefore, we may write equation 8-10 in the form

$$\frac{\partial p}{\partial z} = A_0 + A_1 e^{int} + A_1{*} e^{-int} + A_2 e^{2int} + A_2{*} e^{-2int} \qquad (8\text{-}11)$$

Where we have written A_0 for M_0. Conforming with the representation for the pressure gradient, equation 8-11, we assume that the fluid velocity, w, and the displacement, ξ, of the tube have the form

$$w = w_0 + w_1 e^{int} + w_1{*} e^{-int} + w_2 e^{2int} + w_2{*} e^{-2int} \qquad (8\text{-}12)$$

$$\xi = \xi_0 + \xi_1 e^{int} + \xi_1{*} e^{-int} + \xi_2 e^{2int} + \xi_2{*} e^{-2int} \qquad (8\text{-}13)$$

Expanding the ratio $\bar{w}/c$ as if it is a periodic function, we write

$$\frac{2\xi}{R} = \frac{\bar{w}}{c} \tag{6-33}$$

$$= \bar{w}_0 + \frac{\bar{w}_1}{c_1} e^{int} + \frac{\bar{w}_1^*}{c_1^*} e^{-int} + \frac{\bar{w}_2}{c_2} e^{2int} + \frac{\bar{w}_2^*}{c_2^*} e^{-2int} \tag{8-14}$$

In equation 8-14 we take $\bar{w}_0 = 0$, since the zeroth harmonic (steady state average fluid velocity) will not be affected by the elasticity of the tube wall. With $\bar{w}_0 = 0$, we have from equation 8-14:

$$1 + \frac{2\xi}{R} = 1 + \frac{\bar{w}_1}{c_1} e^{int} + \frac{\bar{w}_1^*}{c_1^*} e^{-int} + \frac{\bar{w}_2}{c_2} e^{2int} + \frac{\bar{w}_2^*}{c_2^*} e^{-2int} \tag{8-15}$$

We recall

$$\bar{w}_1 = \frac{A_1 R^2}{i\mu \alpha_1^2} \left\{ 1 + \eta F_{10}(\alpha_1) \right\} \tag{6-14}$$

and introduce the notation

$$C_1 = \frac{c_0}{c_1} \bar{w}_1 = \left(X_1 - iY_1 \right) \frac{A_1 R^2}{i\mu \alpha_1^2} \left\{ 1 + \eta F_{10}(\alpha_1) \right\}$$

Thus

$$\frac{C_1}{c_0} = \frac{\bar{w}_1}{c_1} = \frac{1}{c_0} \left[\left(X_1 - iY_1 \right) \frac{A_1 R^2}{i\mu \alpha_1^2} \left\{ 1 + \eta F_{10}(\alpha_1) \right\} \right]$$

Therefore, we may write equation 8-15 as

$$1 + \frac{2\xi}{R} = 1 + \frac{C_1}{C_0} e^{int} + \frac{C_1^*}{C_0} e^{-int} + \frac{C_2}{C_0} e^{2int} + \frac{C_2^*}{C_0} e^{-2int}$$

$$= 1 + \frac{1}{C_0}\left[C_1 e^{int} + C_1^* e^{-int} + C_2 e^{2int} + C_2^* e^{-2int} \right] \tag{8-16}$$

where

$$C_m = \left(X_m - iY_m \right) \frac{A_m R^2}{i\mu\alpha_m^2} \left[1 + \eta_m F_{10}(\alpha_m) \right] \tag{8-17}$$

and

$$\frac{c_o}{c_m} = X_m - iY_m$$

Passing to the limiting condition of very stiff constraint, the expression for C_m takes the form

$$C_m = \frac{A_m R^2}{i\mu\alpha_m^2} \left[\frac{3}{4} M'_{10}(\alpha_m) \right]^{1/2} e^{\frac{i}{2}\varepsilon'_{10}(\alpha_m)} \tag{8-18}$$

Note that in passing from the elastic condition (where amplitude and phase is denoted by double prime) to the limiting condition of very stiff constraint (where amplitude and phase is denoted by single prime),

$$\left(\frac{c_o}{c_m}\right)\left|1+\eta F_{10}(\alpha)\right| = \left(\frac{c_o}{c_m}\right)M''_{10}(\alpha) \rightarrow \frac{\sqrt{3}}{2}$$

and phase $\left\{1+\eta F_{10}(\alpha)\right\} = \varepsilon''_{10}(\alpha) \rightarrow \frac{1}{2}\varepsilon'_{10}(\alpha)$

From the above representations for the pressure gradient $\partial p/\partial z$ and the quantity $1+\frac{2\xi}{R}$, we have the following expression for the product $(1+\frac{2\xi}{R})\frac{\partial p}{\partial z}$

$$\left(1+\frac{2\xi}{R}\right)\frac{\partial p}{\partial z} = A_0 + A_1 e^{int} + A_1^* e^{-int} + A_2 e^{2int} + A_2^* e^{-2int}$$

$$+ \frac{A_0}{c_0}\left(C_1 e^{int} + C_1^* e^{-int} + C_2 e^{2int} + C_2^* e^{-2int}\right)$$

$$+ \frac{1}{c_0}\left(A_1 e^{int} + A_1^* e^{-int}\right)\left(C_1 e^{int} + C_1^* e^{-int} + C_2 e^{2int} + C_2^* e^{-2int}\right)$$

$$+ \frac{1}{c_0}\left(A_2 e^{2int} + A_2^* e^{-2int}\right)\left(C_1 e^{int} + C_1^* e^{-int} + C_2 e^{2int} + C_2^* e^{-2int}\right)$$

(8-19)

Now consider the term $\frac{R^2}{\nu}(1 + \frac{2\xi}{R})\frac{\partial w}{\partial t}$, which occurs on the left-hand side of equation 8-3. First note that from

$$w = w_0 + w_1 e^{int} + w_1^* e^{-int} + w_2 e^{2int} + w_2^* e^{-2int}$$

$$\frac{\partial w}{\partial t} = in w_1 e^{int} - in w_1^* e^{-int} + 2in w_2 e^{2int} - 2in w_2^* e^{-2int}$$

Thus

$$\frac{R^2}{\nu}\left(1 + \frac{2\xi}{R}\right)\frac{\partial w}{\partial t} = i\alpha^2\left(w_1 e^{int} - w_1^* e^{-int}\right)$$

$$+ 2i\alpha^2\left(w_2 e^{2int} - w_2^* e^{-2int}\right)$$

$$+ \frac{i\alpha^2}{c_0}\left(w_1 e^{int} - w_1^* e^{-int}\right)\left(c_1 e^{int} + c_1^* e^{-int} + c_2 e^{2int} + c_2^* e^{-2int}\right)$$

$$+ \frac{2i\alpha^2}{c_0}\left(w_2 e^{2int} - w_2^* e^{-2int}\right)\left(c_1 e^{int} + c_1^* e^{-int} + c_2 e^{2int} + c_2^* e^{-2int}\right) \tag{8-20}$$

where we have used $(\frac{R^2}{\nu})(in) = i\alpha^2$.

THE LONGITUDINAL FLUID VELOCITY

Combining equations 8-6, 8-19 and 8-20 and collecting corresponding powers of e^{int}, we obtain a set of equations for w_0, w_1, w_1^*, w_2 and w_2^*. The terms independent of e^{int} give the equation

$$\frac{d^2 w_0}{dy^2} + \frac{1}{y}\frac{dw_0}{dy} = \frac{A_0 R^2}{\mu} + \frac{R^2}{\mu C_0}\left(A_1 C_1^* + A_1^* C_1 + A_2 C_2^* + A_2^* C_2\right)$$

$$+ \frac{i\alpha^2}{C_0}\left\{C_1^* w_1 - C_1 w_1^* + 2\left(C_2^* w_2 - C_2 w_2^*\right)\right\} \tag{8-21}$$

The terms containing e^{int} result in the equation

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = \frac{A_1 R^2}{\mu} + \frac{R^2}{\mu C_0}\left(A_0 C_1 + A_2 C_1^* + A_1^* C_2\right)$$

$$+ \frac{i\alpha^2}{C_0}\left(-C_2 w_1^* + 2C_1^* w_2\right)$$

The terms containing e^{2int} result in the equation

$$\frac{d^2 w_2}{dy^2} + \frac{1}{y}\frac{dw_2}{dy} + 2i^3\alpha^2 w_2 = \frac{A_2 R^2}{\mu} + \frac{R^2}{\mu C_0}\left(A_0 C_2 + A_1 C_1\right)$$

$$+ \frac{i\alpha^2}{C_0} C_1 w_1$$

We may now obtain an approximate solution of equation 8-21, correct to order $1/c_0$ by inserting the known forms for w_1, w_1^*, w_2 and w_2^* on the right-hand side of equation 8-21. If this is done, we can carry through the integration and express the result in terms of functions already known. The first term, A_0R^2/μ, on the right-hand side of equation 8-21 represents the usual Poiseuille solution. Moreover, since equation 8-21 is a linear differential equation, in the unknown w_0, we may consider the contribution to the solution from each term separately. Deleting the Poiseuille term from the right-hand side of equation 8-21, we write the equation for the m^{th} harmonic in the form

$$\frac{d^2 w_0}{dy^2} + \frac{1}{y}\frac{dw_0}{dy} = -\frac{R^2}{c_0\mu}\left(A_m C_m^* + A_m^* C_m\right) + \frac{im\alpha^2}{c_0}\left(C_m^* w_m - C_m w_m^*\right) \tag{8-22}$$

We may, for convenience, introduce the notation

$$(\alpha_m)^2 = m\alpha^2$$

i.e., the value of α corresponding to the m^{th} harmonic is $\alpha_m = (m)^{1/2}\alpha$. We note that equation 8-22 contains only the m^{th} harmonic, $m = 1,2,3,\ldots$. Equation 8-21 contains the first two harmonics. For equation 8-21 to contain all the harmonics, we write it in the form

$$\frac{d^2 w_0}{dy^2} + \frac{1}{y}\frac{dw_0}{dy} = \frac{A_0R^2}{\mu} - \frac{R^2}{c_0\mu}\left(A_1C_1^* + A_1^*C_1 + A_2C_2^* + A_2^*C_2 + A_3C_3^* + A_3^*C_3 + \cdots\right) + \frac{i\alpha^2}{c_0}\left\{C_1^*w_1 - C_1w_1^* + 2\left(C_2^*w_2 - C_2w_2^*\right) + 3\left(C_3^*w_3 - C_3w_3^*\right) + \cdots\right\}$$

Referring to the expression for w, equation 6-2, we write down the representations for w_m and w_m* as

$$w_m = \frac{A_m R^2}{i\mu(m\alpha^2)} \left\{ 1 + \eta_m \frac{J_0(\alpha_m i^{3/2} y)}{J_0(\alpha_m i^{3/2})} \right\} \tag{8-23}$$

$$w_m^* = -\frac{A_m^* R^2}{i\mu(m\alpha^2)} \left\{ 1 + \eta_m^* \frac{J_0(\alpha_m i^{-3/2} y)}{J_0(\alpha_m i^{-3/2})} \right\} \tag{8-24}$$

Inserting these values of w_m and w_m* into the right-hand side of equation 8-22, we obtain

$$\frac{d^2 w_0}{dy^2} + \frac{1}{y}\frac{dw_0}{dy} = \frac{R^2}{c_0\mu} A_m C_m^* \eta_m \frac{J_0(\alpha_m i^{3/2} y)}{J_0(\alpha_m i^{3/2})} - \frac{R^2}{c_0\mu} A_m^* C_m \eta_m^* \frac{J_0(\alpha_m i^{-3/2} y)}{J_0(i^{-3/2}\alpha_m)} \tag{8-25}$$

Note that the constant terms $\frac{A_m R^2}{i\mu(m\alpha^2)}$ and $\frac{A_m{*} R^2}{-i\mu(m\alpha^2)}$ cancel out upon insertion in equation 8-22.

It is known that the differential equation

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + k^2w = 0$$

has the solution $$w = w(y) = J_0(ky)$$

If we set $$ky = (i^{3/2}\alpha_m)y$$

and its conjugate $$(ky)^* = (i^{-3/2}\alpha_m)y$$

and note that a complex function and its conjugate are linearly independent, we may write the solution of equation 8-25 together with the boundary conditions

$$w_0 = 0 \quad \text{at} \quad y = 1$$

$$w_0 < \infty \quad \text{at} \quad y = 0$$

in the form

$$w_o = \frac{R^2}{c_o i^3(\alpha_m)^2\mu} A_m C_m^* \eta_m \left\{1 - \frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\right\}$$

$$+ \frac{R^2}{c_o i^3(\alpha_m)^2\mu} A_m^* C_m \eta_m^* \left\{1 - \frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}\right\}$$

(3-26)

Equation 8-26 may be written as

$$w_o = \frac{R^2}{c_o i (\alpha_m)^2 \mu} A_m C_m^* (-\eta_m) \left\{ 1 - \frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)} \right\}$$

$$- \frac{R^2}{c_o i (\alpha_m)^2 \mu} A_m^* C_m (-\eta_m^*) \left\{ 1 - \frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)} \right\}$$

(8-27)

To obtain the average fluid velocity, we integrate equation 8-27 and then put it in modulus and phase form. We obtain the contribution of the two terms in equation 8-27 to the average velocity

$$\bar{w}_o = \frac{A_m R^2}{i c_o \mu \alpha_m^2} C_m^* (-\eta_m) M_{10}'(\alpha_m) e^{i \epsilon_{10}'(\alpha_m)}$$

$$- \frac{A_m^* R^2}{i c_o \mu \alpha_m^2} C_m (-\eta_m^*) M_{10}'(\alpha_m) e^{-i \epsilon_{10}'(\alpha_m)}$$

(8-28)

or

$$\bar{\omega}_o = \left(\frac{1}{C_o}\right)\frac{A_m R^2}{i\mu\alpha_m^2} C_m^*\left(-\eta_m\right) M_{10}'\left(\alpha_m\right)\left[\cos\varepsilon_{10}'\left(\alpha_m\right) + i\sin\varepsilon_{10}'\left(\alpha_m\right)\right]$$

$$-\left(\frac{1}{C_o}\right)\frac{A_m^* R^2}{i\mu\alpha_m^2} C_m\left(-\eta_m^*\right) M_{10}'\left(\alpha_m\right)\left[\cos\varepsilon_{10}'\left(\alpha_m\right) - i\sin\varepsilon_{10}'\left(\alpha_m\right)\right]$$

(8-29)

Now we insert the expression for C_m and its conjugate, C_m^*, from equation 8-17 into equation 8-29, simplify and obtain for the right-hand side

$$\left(\frac{1}{2C_o}\right)\left(\frac{M_m}{mn\rho}\right)^2\left|X_m - iY_m\right| M_{10}'\left(\alpha_m\right) M_{10}''\left(\alpha_m\right)\cos\Big\{\varepsilon_{10}'\left(\alpha_m\right) - \varepsilon_{10}''\left(\alpha_m\right)$$
$$+ \text{phase}\left(X_m + iY_m\right) + \text{phase}\left(-\eta_m\right)\Big\}$$

(8-30)

For the limiting condition of very stiff constraint, expression 8-30 reduces to the form

$$\left(\frac{1}{2C_o}\right)\left(\frac{M_m}{mn\rho}\right)^2\left(\frac{\sqrt{3}}{2}\right)\left[M_{10}'\left(\alpha_m\right)\right]^{3/2}\cos\frac{\varepsilon_{10}'\left(\alpha_m\right)}{2}$$

(8-31)

CORRECTIONS FOR THE INTERACTIONS BETWEEN HARMONIC COMPONENTS

It is possible to prepare a table of standard correction functions by calculating the factor

$$\frac{\sqrt{3}}{4}\left[M'_{10}(\alpha_m)\right]^{3/2}\cos\frac{\epsilon'_{10}(\alpha_m)}{2}$$

from equation 8-31 for the full range of values of α. We denote this factor by $E(m,-m)$

$$E(m,-m) = \frac{\sqrt{3}}{4}\left[M'_{10}(\alpha_m)\right]^{3/2}\cos\frac{\epsilon'_{10}(\alpha_m)}{2} \tag{8-32}$$

$E(m,-m)$ represents the standard correction function for finite expansion of the tube, expressing the effect of the m^{th} harmonic on the steady flow. Accordingly, for any given pulse form the correction to the steady term is obtained from equation 8-31 by summing over all the harmonics. Note that

$$\left(\frac{1}{2C_0}\right)\left(\frac{M_m}{mn\rho}\right)^2\left(\frac{\sqrt{3}}{2}\right)\left[M'_{10}(\alpha_m)\right]^{3/2}\cos\frac{\epsilon'_{10}(\alpha_m)}{2}$$

$$= \left(\frac{1}{C_0}\right)\left(\frac{M_m}{mn\rho}\right)^2 E(m,-m)$$

Therefore, the correction to the steady term is

$$\left(\frac{1}{C_o}\right) \sum_m \left(\frac{M_m}{mn\rho}\right)^2 E(m, -m) \tag{8-33}$$

This correction will be in the same direction as the steady stream.

We now turn to the construction of the corresponding standard correction functions denoted as follows.

$E(\ell,m)$: standard correction function for finite expansion of the tube, expressing the effect on the average velocity of the $(\ell+m)$th harmonic of interaction between the ℓth and mth harmonics.

We recall the equation for the mth harmonic

$$\frac{d^2 w_o}{dy^2} + \frac{1}{y}\frac{dw_o}{dy} = -\frac{R^2}{C_o\mu}\left(A_m C_m^* + A_m^* C_m\right) + \frac{im\alpha^2}{C_o}\left(C_m^* w_m - C_m w_m^*\right) \tag{8-22}$$

We may write the left-hand side of equation 8-22 as

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} - i(-m+m)\alpha^2 w$$

The equation describing the correction for the cross-effects between the ℓth and mth harmonics is

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3(\ell+m)\alpha^2 w = -\frac{R^2}{C_o\mu}\left(A_m C_\ell + A_\ell C_m\right)$$

$$+ \frac{i\alpha^2}{C_o}\left(m C_\ell w_m + \ell C_m w_\ell\right)$$

or

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} - i(\ell + m)\alpha^2 w = \frac{-R^2}{c_0\mu}(A_m C_\ell + A_\ell C_m) + \frac{i\alpha^2}{c_0}(mC_\ell w_m + \ell C_m w_\ell) \tag{8-34}$$

Inserting the values of w_m and w_ℓ, given by

$$w_m = \frac{A_m R^2}{i\mu m\alpha^2}\left\{1 + \eta_m \frac{J_0(i^{3/2}\alpha_m y)}{J_0(i^{3/2}\alpha_m)}\right\}$$

$$w_\ell = \frac{A_\ell R^2}{i\mu \ell\alpha^2}\left\{1 + \eta_\ell \frac{J_0(i^{3/2}\alpha_\ell y)}{J_0(i^{3/2}\alpha_\ell)}\right\}$$

into equation 8-34, we obtain

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} - i(\ell+m)\alpha^2 w = -\frac{R^2}{c_0\mu}\left(A_m C_\ell + A_\ell C_m\right)$$

$$+ \frac{i\alpha^2}{c_0}\left[mC_\ell \frac{A_m R^2}{i\mu m\alpha^2}\left\{1 + \eta_m \frac{J_0(i^{3/2}\alpha_m y)}{J_0(i^{3/2}\alpha_m)}\right\}\right.$$

$$\left. + \ell C_m \frac{A_\ell R^2}{i\mu \ell\alpha^2}\left\{1 + \eta_\ell \frac{J_0(i^{3/2}\alpha_\ell y)}{J_0(i^{3/2}\alpha_\ell)}\right\}\right]$$

or

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} - i(\ell + m)\alpha^2 w = \frac{R^2}{c_o \mu} A_m C_\ell \eta_m \frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}$$

$$+ \frac{R^2}{c_o \mu} A_\ell C_m \eta_\ell \frac{J_o(i^{3/2}\alpha_\ell y)}{J_o(i^{3/2}\alpha_\ell)}$$

(8-35)

Now we write down a well-known result. The solution of the differential equation

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + k^2 w = \frac{AJ_0(\ell y)}{J_0(\ell)}$$

which satisfies the boundary conditions

$$w = 0 \text{ at } y = 1$$

$$w < \infty \text{ at } y = 0$$

is given by

$$w(y) = \frac{A}{k^2 - \ell^2}\left[\frac{J_o(\ell y)}{J_o(\ell)} - \frac{J_o(ky)}{J_o(k)}\right] \tag{8-36}$$

This solution is valid for $k \neq \ell$.

In the above, when we impose the condition that all the correction terms vanish at the tube wall, $y = 1$, we are in effect making a further approximation. Physically, it enforces the condition that the motion of the wall is due to the main terms only and the correction terms have no effect. Since

the correction terms are small, this approximation may be adequate. For complete consistency, the arbitrary constants in the expressions for average velocity which are substituted in the equation should be left "floating" and the fully corrected solution substituted back in the equations of motion of the tube. The frequency equation would then be nonlinear and the pulse velocity would depend on the particular form of the pressure function. The same situation will arise if a similar method is used to calculate the inertia term correction.

The solution of equation 8-35 according to equation 8-36 may be written as

$$w = \frac{1}{i^3\left[(\ell+m)-m\right]\alpha^2}\left(\frac{1}{c_0}\right)C_\ell A_m \eta_m$$

$$\left[1 - \frac{J_0\left(i^{3/2}\alpha_{\ell+m}y\right)}{J_0\left(i^{3/2}\alpha_{\ell+m}\right)} - \left\{1 - \frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right\}\right]$$

$$+ \frac{1}{i^3\left[(m+\ell)-\ell\right]\alpha^2}\left(\frac{1}{c_0}\right)C_\ell A_m \eta_m$$

$$\left[1 - \frac{J_0\left(i^{3/2}\alpha_{m+\ell}y\right)}{J_0\left(i^{3/2}\alpha_{m+\ell}\right)} - \left\{1 - \frac{J_0\left(i^{3/2}\alpha_\ell y\right)}{J_0\left(i^{3/2}\alpha_\ell\right)}\right\}\right] \tag{8-37}$$

The corresponding average velocity (corresponding to equation 8-37) is obtained according to

$$\bar{w} = \int_0^1 w\,(2y)\,dy$$

Thus

$$\bar{w} = \left(\frac{R^2}{c_o\mu}\right) C_\ell A_m \left(-\eta_m\right)\left(\frac{1}{i\ell\alpha^2}\right)\int_0^1 \left[1 - \frac{J_o\left(i^{3/2}\alpha_{\ell+m}y\right)}{J_o\left(i^{3/2}\alpha_{\ell+m}\right)}\right](2y)\,dy$$

$$-\int_0^1 \left[1 - \frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right](2y)\,dy \tag{8-38}$$

+ the preceding expression with ℓ and m interchanged.

If we use the notation

$$M'_{10}\left(\alpha_{\ell+m}\right) = \left|1 - \frac{2J_1\left(i^{3/2}\alpha_{\ell+m}\right)}{i^{3/2}\alpha_{\ell+m}J_o\left(i^{3/2}\alpha_{\ell+m}\right)}\right|$$

$$\varepsilon'_{10}\left(\alpha_{\ell+m}\right) = \text{phase}\left\{1 - \frac{2J_1\left(i^{3/2}\alpha_{\ell+m}\right)}{i^{3/2}\alpha_{\ell+m}J_0\left(i^{3/2}\alpha_{\ell+m}\right)}\right\}$$

then in amplitude and phase form, we may write equation 8-38 as

$$\bar{w} = \left(\frac{R^2}{c_o\mu}\right)C_\ell A_m\left(-\eta_m\right)\left(\frac{1}{i\ell\alpha^2}\right)\left\{M'_{10}\left(\alpha_{\ell+m}\right)e^{i\varepsilon'_{10}\left(\alpha_{\ell+m}\right)} - M'_{10}\left(\alpha_m\right)e^{i\varepsilon'_{10}\left(\alpha_m\right)}\right\}$$

+ preceding expression with ℓ and m interchanged, i.e.,

$$\bar{w} = \left(\frac{R^2}{c_o\mu}\right)C_\ell A_m\left(-\eta_m\right)\left(\frac{1}{i\ell\alpha^2}\right)\left\{M'_{10}\left(\alpha_{\ell+m}\right)e^{i\varepsilon'_{10}\left(\alpha_{\ell+m}\right)} - M'_{10}\left(\alpha_m\right)e^{i\varepsilon'_{10}\left(\alpha_m\right)}\right\}$$

$$+\left(\frac{R^2}{c_o\mu}\right)C_m A_\ell\left(-\eta_\ell\right)\left(\frac{1}{im\alpha^2}\right)\left\{M'_{10}\left(\alpha_{m+\ell}\right)e^{i\varepsilon'_{10}\left(\alpha_{m+\ell}\right)} - M'_{10}\left(\alpha_\ell\right)e^{i\varepsilon'_{10}\left(\alpha_\ell\right)}\right\}$$

(8-39)

This is the correction factor for $\bar{w}$ being the effect of the ℓ^{th} and m^{th} harmonics on the $(\ell+m)^{th}$ harmonic.

We will write this correction factor in more compact form by introducing the following notation.

$$F(\ell,m) = \left(\frac{m}{\ell}\right)\left(X_\ell - iY_\ell\right) M''_{10}(\alpha_\ell) e^{i\varepsilon''_{10}(\alpha_\ell)} \cdot$$

$$\left\{ M'_{10}(\alpha_{\ell+m}) e^{i\varepsilon'_{10}(\alpha_{\ell+m})} - M'_{10}(\alpha_m) e^{i\varepsilon'_{10}(\alpha_m)} \right\} \tag{8-40}$$

$$F(m,\ell) = \left(\frac{\ell}{m}\right)\left(X_m - iY_m\right) M''_{10}(\alpha_m) e^{i\varepsilon''_{10}(\alpha_m)} \cdot$$

$$\left\{ M'_{10}(\alpha_{m+\ell}) e^{i\varepsilon'_{10}(\alpha_{m+\ell})} - M'_{10}(\alpha_\ell) e^{i\varepsilon'_{10}(\alpha_\ell)} \right\}$$

$$E(\ell,m) = \left(-\eta_m\right) F(\ell,m) + \left(-\eta_\ell\right) F(m,\ell) \tag{8-41}$$

$$C_m = \left(X_m - iY_m\right)\frac{A_m R^2}{i\mu(\alpha_m^2)}\left\{1 + \eta_m F_{10}(\alpha_m)\right\}$$

$$= \left(X_m - iY_m\right)\frac{A_m R^2}{i\mu(m\alpha^2)}\left\{M_{10}''(\alpha_m)e^{i\varepsilon_{10}''(\alpha_m)}\right\}$$

$$C_\ell = \left(X_\ell - iY_\ell\right)\frac{A_\ell R^2}{i\mu(\alpha_\ell^2)}\left\{1 + \eta_\ell F_{10}(\alpha_\ell)\right\}$$

$$= \left(X_\ell - iY_\ell\right)\frac{A_\ell R^2}{i\mu(\ell\alpha^2)}\left\{M_{10}''(\alpha_\ell)e^{i\varepsilon_{10}''(\alpha_\ell)}\right\}$$

After some simplification, the right-hand side of equation 8-39 may be written in the form

$$\left(\frac{-1}{C_0}\right)\left(\frac{A_m R^2}{\mu m\alpha^2}\right)\left(\frac{A_\ell R^2}{\mu \ell\alpha^2}\right)\left[\left(-\eta_m\right)F(\ell,m) + \left(-\eta_\ell\right)F(m,\ell)\right]$$

or

$$\left(\frac{1}{C_0}\right)\left(\frac{A_m R^2}{\mu m\alpha^2}\right)\left(\frac{A_\ell R^2}{\mu \ell\alpha^2}\right)\left[E(\ell,m)\right]$$

We will now consider the actual (real) form of the $(\ell+m)^{th}$ harmonic. If the pressure gradient is in real form, then the real form for the average longitudinal fluid velocity corresponding to the $(\ell+m)^{th}$ harmonic is obtained by writing the sum of the complex velocity and its conjugate as follows:

$$\bar{w}_{\ell+m} e^{i(\ell+m)nt} + \bar{w}^*_{\ell+m} e^{-i(\ell+m)nt}$$

We note that in equation 8-39 the factor $R^2/\mu\alpha^2$ may be written as

$$\frac{R^2}{\mu\alpha^2} = \frac{\nu}{\mu n} = \frac{1}{\rho n}$$

Moreover, the factors A_m and A_ℓ appearing in equation 8-39, according to earlier defined notation, may be written as

$$A_m = \frac{1}{2} M_m \; e^{i\phi_m}$$

$$A_\ell = \frac{1}{2} M_\ell \; e^{-i\phi_\ell}$$

since A_m and A_ℓ are complex quantities.

Thus, when considering only the real parts for the correction to the real quantity

$$\bar{w}_{\ell+m} e^{i(\ell+m)nt} + \bar{w}^*_{\ell+m} e^{-i(\ell+m)nt}$$

corresponding to the earlier correction

$$\left(-\frac{1}{c_0}\right)\left(\frac{A_m R^2}{\mu m \alpha^2}\right)\left(\frac{A_\ell R^2}{\mu \ell \alpha^2}\right) E(\ell, m)$$

we have the correction term

$$\left(-\frac{1}{c_0}\right)\left(\frac{M_m}{m n \rho_0}\right)\left(\frac{M_\ell}{\ell n \rho_0}\right)\left|E(\ell,m)\right| \cos\left\{(\ell+m)nt + \varphi_\ell + \varphi_m + \text{phase } E(\ell,m)\right\}$$

(8-42)

Note that

$$A_m = \frac{1}{2} M_m e^{i\varphi_m}$$

$$A_\ell = \frac{1}{2} M_\ell e^{i\varphi_\ell}$$

$$E(\ell,m) = \left|E(\ell,m)\right| \text{phase}\left\{E(\ell,m)\right\}$$

In order to evaluate the correction corresponding to the $(\ell+m)^{th}$ harmonic, it is convenient to have a table of $E(\ell,m)$ in modulus and phase form.

In the elastic case, the formula for $E(\ell,m)$ is

$$E(\ell,m) = (-\eta_m)[F(\ell,m)] + (-\eta_\ell)[F(m,\ell)] \qquad (8\text{-}41)$$

For the limiting condition of very stiff constraint, since

$$\eta_m = -1$$

and

$$\eta_\ell = -1 \; ,$$

the above formula for $E(\ell,m)$ reduces to the form

$$E(\ell,m) = F(\ell,m) + F(m,\ell)$$

In the elastic case, the formula for $F(\ell,m)$ is

$$F(\ell,m) = \left(\frac{m}{\ell}\right)\left(X_\ell - iY_\ell\right) M''_{10}(\alpha_\ell)\, e^{i\varepsilon''_{10}(\alpha_\ell)} \cdot \left\{ M'_{10}(\alpha_{\ell+m})\, e^{i\varepsilon'_{10}(\alpha_{\ell+m})} - M'_{10}(\alpha_m)\, e^{i\varepsilon'_{10}(\alpha_m)} \right\} \tag{8-40}$$

For the limiting condition of very stiff constraint, since

$$\frac{c_c}{c_\ell} M''_{10}(\alpha_\ell) \rightarrow \frac{\sqrt{3}}{2}\left[M'_{10}(\alpha_\ell)\right]^{1/2}$$

and

$$\varepsilon''_{10}(\alpha_\ell) \rightarrow \frac{1}{2}\,\varepsilon'_{10}(\alpha_\ell)$$

the above formula for $F(\ell,m)$ for the elastic case becomes

$$F(\ell,m) = \frac{m}{\ell}\left[\frac{3}{4}M'_{10}(\alpha_\ell)\right]^{1/2} e^{\frac{i}{2}\varepsilon'_{10}(\alpha_\ell)}$$

$$\left\{M'_{10}(\alpha_{\ell+m})e^{i\varepsilon'_{10}(\alpha_{\ell+m})} - M'_{10}(\alpha_m)e^{i\varepsilon'_{10}(\alpha_m)}\right\}$$

(8-43)

The formula for $F(\ell,-m)$ can be written down at once by substituting $-m$ for m in equation 8-43. Thus

$$F(\ell,-m) = -\frac{m}{\ell}\left[\frac{3}{4}M'_{10}(\alpha_\ell)\right]^{1/2} e^{\frac{i}{2}\varepsilon'_{10}(\alpha_\ell)}$$

$$\cdot\left\{M'_{10}(\alpha_{\ell-m})e^{i\varepsilon'_{10}(\alpha_{\ell-m})} - M'_{10}(\alpha_m)e^{-i\varepsilon'_{10}(\alpha_m)}\right\}$$

(8-44)

The above formulas for the corrections do not apply when either $\ell = 0$ or $m = 0$. This can be seen from equation 8-37. In equation 8-37, note that when $\ell = 0$, the factor $\frac{1}{[(\ell+m)-m]}$ in the denominator of the first term will be zero. Similarly, when $m = 0$, the factor $\frac{1}{[(m+\ell)-\ell]}$ in the denominator of the second term is zero.

The equation describing the effect on the mth harmonic of its own interaction has the form

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3 m \alpha^2 w = -\frac{R^2}{\mu c_0} A_0 C_m \tag{8-45}$$

Since the right-hand side is a constant, in analogy with the solution of equation 2-11 we write the solution of equation 8-45 as

$$w = \frac{A_0 R^2 C_m}{i c_0 \mu m \alpha^2}\left\{1 - \frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right\}$$

Proceeding in the same manner as we did for obtaining the correction term (8-42) for the solution of equation 8-35, we obtain the correction for the solution of equation 8-45 in the form

$$-\frac{1}{c_0}\left(\frac{M_0}{mn\rho_0}\right)\left(\frac{M_m}{mn\rho_0}\right)\left|E(m,0)\right| \cos\left\{mnt + \phi_m + \text{phase}\left[E(m,0)\right]\right\} \tag{8-45'}$$

where

$$E(m,0) = \left(X_m - i Y_m\right) M''_{10}(\alpha_r)\, e^{i\epsilon''_{10}(\alpha_m)} \qquad M'_{10}(\alpha_m)\, e^{i\epsilon'_{10}(\alpha_m)} \tag{8-46}$$

The right side of equation 8-46 is obtained in the same manner as $E(\ell,m)$ was obtained for the solution of equation 8-35.

For the limiting condition of very stiff constraint, the expression for E(m,0) reduces to the form

$$E(m,0) = \frac{\sqrt{3}}{2}\left[M'_{10}(\alpha_m)\right]^{1/2} e^{\frac{i}{2}\varepsilon'_{10}(\alpha_m)} \quad M'_{10}(\alpha_m) e^{i\varepsilon'_{10}(\alpha_m)}$$

$$= \frac{\sqrt{3}}{2}\left[M'_{10}(\alpha_m)\right]^{3/2} e^{\frac{3i}{2}\varepsilon'_{10}(\alpha_m)} \qquad (8-47)$$

APPLICATION TO ARTERIAL FLOW

We now consider the practical application of these formulas. In the application to arterial flow, four harmonics in the pressure gradient are usually sufficient. Taking account of four harmonics, from equation 8-23 we write the correction to the steady term of the axial velocity as

$$\frac{1}{C_0}\sum_{m=1}^{4}\left(\frac{M_m}{mn\rho}\right)^2 E(m,-m)$$

Since the steady term is $\frac{A_0 R^2}{\mu}$, we write the corrected average fluid velocity as

$$\bar{w} = \frac{A_0 R^2}{\mu} + \frac{1}{C_0}\sum_{m=1}^{4}\left(\frac{M_m}{mn\rho_0}\right)^2 E(m,-m)$$

or

$$\bar{w} = \frac{M_0 R^2}{\mu} + \frac{1}{C_0}\sum_{m=1}^{4}\left(\frac{M_m}{mn\rho_0}\right)^2 E(m,-m) \qquad (8-48)$$

If we denote the pressure gradient in real form by $M_1 \cos(nt - \phi_1)$ then the first harmonic of the average velocity corresponding to this pressure gradient is, according to equation 6-15

$$\bar{w}_1 = \frac{M_1 R^2}{\mu \alpha^2} M''_{10}(\alpha_1) \sin\left(nt - \varphi_1 + \varepsilon''_{10}(\alpha_1)\right)$$

or

$$\bar{w}_1 = \frac{M_1 M''_{10}(\alpha_1)}{n \rho_0} \sin\left(nt - \varphi_1 + \varepsilon''_{10}(\alpha_1)\right)$$

The correction to this first harmonic of the average velocity, $\bar{w}_1$, is, according to equation 8-45', given by

$$-\frac{1}{c_0}\left(\frac{M_0}{n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(1,0)\right| \cos\left\{nt - \varphi_1 + \text{phase } E(1,0)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_2}{2n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(2,-1)\right| \cos\left\{nt - (\varphi_2 - \varphi_1) + \text{phase } E(2,-1)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_3}{3n\rho_0}\right)\left(\frac{M_2}{2n\rho_0}\right)\left|E(3,-2)\right| \cos\left\{nt - (\varphi_3 - \varphi_2) + \text{phase } E(3,-2)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_4}{4n\rho_0}\right)\left(\frac{M_3}{3n\rho_0}\right)\left|E(4,-3)\right| \cos\left\{nt - (\varphi_4 - \varphi_3) + \text{phase } E(4,-3)\right\}$$

Similarly, the second harmonic of the average fluid velocity corresponding to a pressure gradient in the form $M_2 \cos(2nt - \phi_2)$ is given by

$$\bar{w}_2 = \frac{M_2}{2n\rho_0} M''_{10}(\alpha_2) \sin\left\{2nt - \varphi_2 + \varepsilon''_{10}(\alpha_2)\right\}$$

and the correction to this second harmonic, $\bar{w}_2$, is

$$-\frac{1}{c_0}\left(\frac{M_0}{2n\rho_0}\right)\left(\frac{M_2}{2n\rho_0}\right)\left|E(2,0)\right| \cos\left\{2nt - \varphi_2 + \text{phase } E(2,0)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_1}{n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(1,1)\right| \cos\left\{2nt - (\varphi_1 + \varphi_1) + \text{phase } E(1,1)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_3}{3n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(3,-1)\right| \cos\left\{2nt - (\varphi_3 - \varphi_1) + \text{phase } E(3,-1)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_4}{4n\rho_0}\right)\left(\frac{M_2}{2n\rho_0}\right)\left|E(4,-2)\right| \cos\left\{2nt - (\varphi_4 - \varphi_2) + \text{phase } E(4,-2)\right\}$$

The third harmonic of the average fluid velocity corresponding to a pressure gradient in the form $M_3 \cos(3nt - \phi_3)$ is given by

$$\bar{w}_3 = \frac{M_3}{3n\rho_0} M''_{10}(\alpha_3) \sin\left\{3nt - \varphi_3 + \varepsilon''_{10}(\alpha_3)\right\}$$

and the correcti'n to this is

$$-\frac{1}{c_0}\left(\frac{M_0}{3n\rho_0}\right)\left(\frac{M_3}{3n\rho_0}\right)\left|E(3,0)\right| \cos\left\{3nt - \varphi_3 + \text{phase } E(3,0)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_2}{2n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(2,1)\right| \cos\left\{3nt - (\varphi_1 + \varphi_2) + \text{phase } E(2,1)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_4}{4n\rho_0}\right)\left(\frac{M_1}{n\rho_0}\right)\left|E(4,-1)\right| \cos\left\{3nt - (\varphi_4 - \varphi_1) + \text{phase } E(4,-1)\right\}$$

The fourth harmonic of the average fluid velocity corresponding to a pressure gradient in the form $M_4 \cos(4nt - \phi_4)$ is given by

$$\bar{w}_4 = \frac{M_4}{4n\rho_0} M''_{10}(\alpha_4) \sin\left\{4nt - \varphi_4 + \varepsilon''_{10}(\alpha_4)\right\}$$

and the correction to this is

$$-\frac{1}{c_0}\left(\frac{M_0}{4n\rho_0}\right)\left(\frac{M_4}{4n\rho_0}\right)\left|E(4,0)\right| \cos\left\{4nt - \varphi_4 + \text{phase } E(4,0)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_1}{n\rho_0}\right)\left(\frac{M_3}{3n\rho_0}\right)\left|E(3,1)\right| \cos\left\{4nt - (\varphi_1 + \varphi_3) + \text{phase } E(3,1)\right\}$$

$$-\frac{1}{c_0}\left(\frac{M_2}{2n\rho_0}\right)\left(\frac{M_2}{2n\rho_0}\right)\left|E(2,2)\right| \cos\left\{4nt - (\varphi_2 + \varphi_2) + \text{phase } E(2,2)\right\}$$

As an example of the magnitude of a typical set of corrections, a complete calculation has been done for one of McDonald's experiments (Womersley, 1954) on the femoral artery of the dog. Fourier analysis of the pressure gradient record gave

$$
\begin{aligned}
-\frac{\partial p}{\partial z} &= 0.159 + 0.774 \cos (nt + 0^\circ\ 39') \\
&+ 1.317 \cos (2nt - 22^\circ\ 45') - 0.743 \cos (3nt + 26^\circ\ 30') \\
&- 0.414 \cos (4nt - 16^\circ\ 39')
\end{aligned}
$$

These coefficients are in millimeters of mercury per centimeter. The conversion constant, to bring them to absolute units, was included in the common factor $1/n\rho_0$. It is not possible to make an accurate estimate of c_0 until accurate measurements of the pulse velocity have been made over short lengths of artery, say 3 or 4 cm, over which the diameter is reasonably constant. A rough estimate of the experimental observations gave a maximum ξ/R of about 6%. Since $\left|\frac{2\xi}{R}\right| = \frac{|\bar{w}|}{c_0} |X - iY|$ and the maximum average velocity was 88 cm/sec, this suggests $600 < c_0 < 700$ cm/sec. The pulse velocity, estimated from records taken on other experiments, suggested a value of c_0 of about 850 cm/sec. Two sets of corrections have, therefore, been calculated for $c_0 = 1000$ cm/sec and for $c_0 = 500$ cm/sec. These have been carried out for the limiting condition of stiff constraint only. According to figure 53, the effect of the correction is not very marked, even for $c_0 = 500$ cm/sec. The curve for $c_0 = 1000$ is not shown. In table VII the Fourier coefficients are shown for the uncorrected average velocity and the two sets of corrections.

TABLE VII

Values of the Fourier Coefficients for the Calculation of the Average Velocity, With and Without the Nonlinear Correction for Finite Expansion

	Uncorrected		Corrected $c_0 = 1000$		Corrected $c_0 = 500$	
m	Coeff. of cos mnt	Coeff. of sin mnt	Coeff. of cos mnt	Coeff. of sin mnt	Coeff. of cos mnt	Coeff. of sin mnt
1	+19.08	+33.14	20.01	32.44	20.94	31.75
2	-31.78	+14.89	-32.57	15.57	-33.37	16.25
3	-8.79	-10.58	-8.47	-10.69	-8.16	-10.79
4	-0.44	-5.86	-0.15	-5.73	0.14	-5.47

These results indicate that, particularly during systolic flow, the main effect of the finite expansion of the tube on the rate of flow is the factor $(1 + \frac{2\xi}{R})$ when multiplying the average velocity by the cross-sectional area. See equation 8-2.

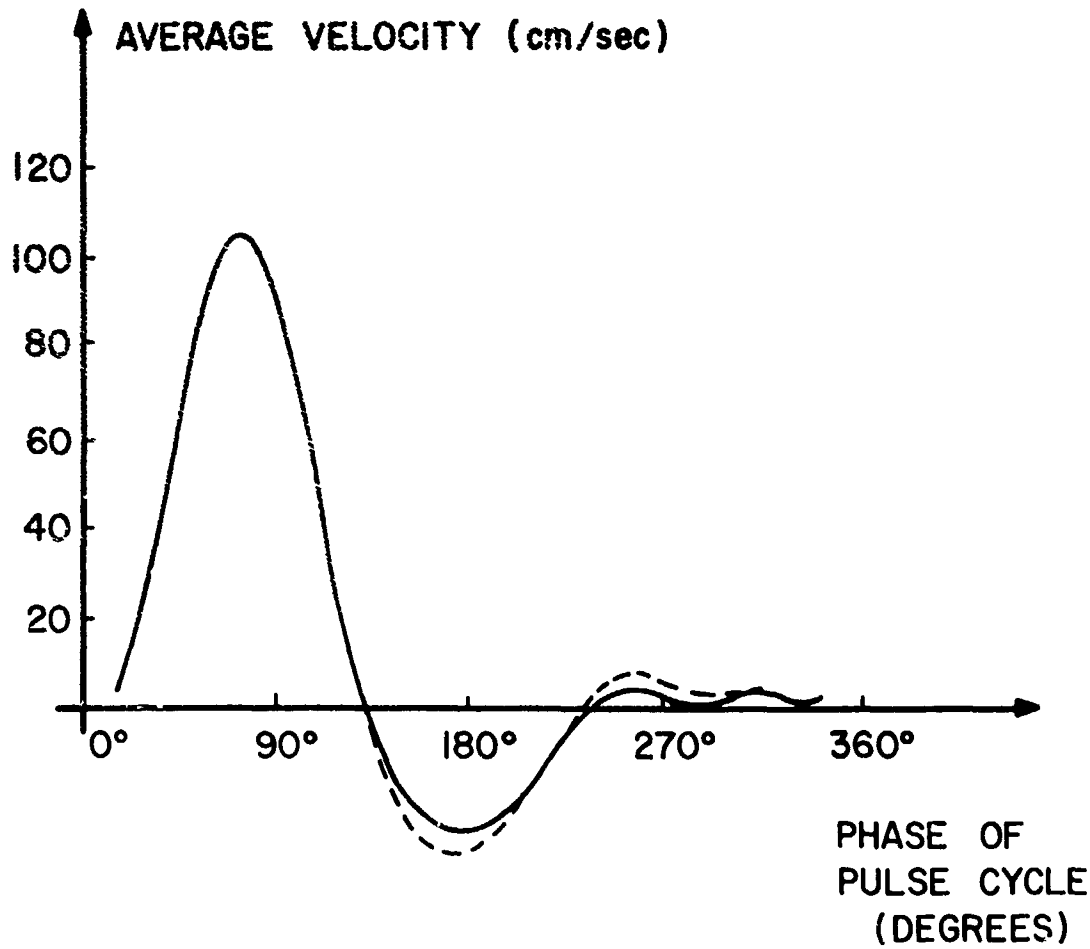

Figure 53. Variation in average velocity over one cycle in the femoral artery of the dog, calculated from the observed pressure gradient of figure 14. FULL LINE, without expansion correction; BROKEN LINE, with expansion correction and c_0 = 500 cm/sec.

SECTION IX

CORRECTIONS FOR THE QUADRATIC TERMS IN THE EQUATIONS OF VISCOUS FLUID MOTION

INTRODUCTION

We start with the Stokes stream function and obtain a general result which is used for the solution of the equations of fluid motion. As in the preceding section, we combine the harmonic representations of the pressure gradient and the longitudinal and radial fluid velocities with the equation describing the fluid motion and obtain the harmonic components of the longitudinal fluid velocity. We next determine the interactions between these harmonic components and obtain the standard correction function for the effect of the quadratic terms in the equations of motion.

A GENERAL RESULT

The correction for the longitudinal velocity due to the quadratic terms in the Navier-Stokes equations follows the same pattern as the correction for the longitudinal velocity due to finite expansion of the tube discussed in section VIII. However, there is one important difference which is indicated below.

Consider the equation for the longitudinal velocity

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial z^2} \tag{3-11}$$

If we neglect the term $\partial^2 w/\partial z^2$, since it is of order n^2R^2/c^2, and write $y = r/R$, we obtain

$$\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{R^2}{\nu}\frac{\partial w}{\partial t} = \frac{R^2}{\mu}\frac{\partial p}{\partial z} + \frac{R}{\nu}u\frac{\partial w}{\partial y} + \frac{R^2}{\nu}w\frac{\partial w}{\partial z} \tag{9-1}$$

If we substitute the same forms for p and w (as in section VIII) into equation 9-1, and seek a solution, we find that the functions on the right-hand side of the resulting equations are (since they contain quadratic terms in the velocity components) products of three Bessel functions. When we try to solve these by the method of variation of parameters, the resulting quadratures involve products of three Bessel functions which do not reduce to known forms. To find the average velocity across the tube requires a further quadrature and the amount of numerical integration required is, at first sight, quite formidable.

Our main objective is to find the effect of the quadratic terms, $u \frac{\partial w}{\partial y}$ and $w \frac{\partial w}{\partial z}$, appearing in equation 9-1, on the average longitudinal velocity. Thus, there is an obvious advantage in seeking a method of solution of equation 9-1 which will give the average longitudinal velocity directly without the calculation of the velocity profile, $w = w(y)$, across the tube. This direct method of calculating the longitudinal velocity consists in using the quantity defined by

$$q = \int_0^y w(2y)\,dy \tag{9-2}$$

which is in effect Stokes' stream function of the fluid motion.

Before deriving the detailed equations from equation 9-1, we prove a general result which will be required for their solution. In analogy with equation 2-4, consider the equation

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3\alpha^2 w = f(y) \tag{9-3}$$

where $f(y)$ is a known function of y. The corresponding equation for q is

$$\frac{d^2q}{dy^2} - \frac{1}{y}\frac{dq}{dy} + i^3\alpha^2 q = g(y) \tag{9-4}$$

where $$g(y) = \int_0^y f(y)(2y)\,dy\ .$$

To show this correspondence, multiply equation 9-3 through by $2y$ and integrate with respect to y. We obtain

$$\int_0^y 2y\,\frac{d^2w}{dy^2}\,dy + \int_0^y 2\,\frac{dw}{dy}\,dy + \int_0^y i^3\alpha^2 w(2y)\,dy = g(y) \tag{9-5}$$

According to equation 9-2, the third term on the left-hand side of equation 9-5 may be written as $(i^3\alpha^2)q$.

Moreover, from $q = \int_0^y w(2y)\,dy$

we have $$\frac{dq}{dy} = w(2y)$$

and $$\frac{d^2q}{dy^2} = 2y\,\frac{dw}{dy} + 2w\ .$$

Furthermore, $\frac{1}{y}\frac{dq}{dy} = 2w$

Thus, the first two terms on the left side of equation 9-4 may be written as

$$\frac{d^2q}{dy^2} - \frac{1}{y}\frac{dq}{dy} = 2y\frac{dw}{dy} + 2w - 2w = 2y\frac{dw}{dy} \tag{9-6}$$

According to integration by parts,

$$\int_0^y \frac{d^2w}{dy^2}(2y)\,dy = \frac{dw}{dy}(2y) - \int_0^y 2\frac{dw}{dy}\,dy$$

or

$$2y\frac{dw}{dy} = \int_0^y \frac{d^2w}{dy^2}(2y)\,dy + \int_0^y 2\frac{dw}{dy}\,dy \tag{9-7}$$

From equations 9-6 and 9-7 we note that

$$\frac{d^2q}{dy^2} - \frac{1}{y}\frac{dq}{dy} = \int_0^y \frac{d^2w}{dy^2}(2y)\,dy + \int_0^y 2\frac{dw}{dy}\,dy = 2y\frac{dw}{dy}$$

We wish to obtain a solution of equation 9-4. One boundary condition, namely, q=0 at y=0, is built into the definition of q:

$$q = \int_0^y w(2y)\,dy$$

So the solution of equation 9-4 will be obtained in terms of one arbitrary constant, A.

The solution of equation 9-4 under the condition q=0 at y=0 is

$$q = Ay\,J_1(i^{3/2}\alpha y) + y\,J_1(i^{3/2}\alpha y)\int_0^y \left[y\left\{J_1(i^{3/2}\alpha y)\right\}^2\right]^{-1} dy \cdot \int_0^y g(y)\,J_1(i^{3/2}\alpha y)\,dy \tag{9-8}$$

Imposing the condition of no slip at the tube wall, w = 0 at y = 1, the condition for determining the value of A in equation 9-8 becomes

$$\frac{dq}{dy} = 0 \quad \text{at} \quad y = 1 \tag{9-9}$$

This follows from differentiating equation 9-2 with respect to y and setting w = 0. Differentiating the expression for q(y) in equation 9-8 and inserting y = 1, we find

$$\left.\frac{dq}{dy}\right]_{y=1} = A\,i^{3/2}\alpha\,J_0(i^{3/2}\alpha)$$
$$+\ i^{3/2}\alpha\,J_0(i^{3/2}\alpha)\int_0^1\left[y\left\{J_1(i^{3/2}\alpha y)\right\}^2\right]^{-1}dy\int_0^y g(y)\,J_1(i^{3/2}\alpha y)\,dy$$
$$+\ \frac{1}{J_1(i^{3/2}\alpha)}\int_0^1 g(y)\,J_1(i^{3/2}\alpha y)\,dy = 0 \tag{9-10}$$

Solving for A, we have

$$A = -\frac{i^{3/2}\alpha J_0\left(i^{3/2}\alpha\right)}{i^{3/2}\alpha J_0\left(i^{3/2}\alpha\right)} \int_0^1 \frac{1}{y\left[J_1\left(i^{3/2}\alpha y\right)\right]^2}\, dy \int_0^y g(y)\, J_1\left(i^{3/2}\alpha y\right) dy$$

$$- \frac{1}{i^{3/2}\alpha J_0\left(i^{3/2}\alpha\right) J_1\left(i^{3/2}\alpha\right)} \int_0^1 g(y)\, J_1\left(i^{3/2}\alpha y\right) dy$$

We may write the expression for A in the form

$$A = -\int_0^1 X dy \int_0^y Y dy - \frac{1}{B}\int_0^1 Y dy$$

Substituting this value of A into equation 9-8 we obtain:

$$q(y) = -y\, J_1\left(i^{3/2}\alpha y\right)\left\{\int_0^1 X\, dy \int_0^y Y\, dy + \frac{1}{B}\int_0^1 Y\, dy\right\}$$

$$+\, y\, J_1\left(i^{3/2}\alpha y\right)\int_0^y X\, dy \int_0^y Y\, dy$$

$$q(y) = y J_1(i^{3/2}\alpha y)\left\{-\int_0^1 X\,dy \int_0^y Y\,dy\right.$$

$$\left. + \int_0^y X\,dy \int_0^y Y\,dy \right\} - y J_1(i^{3/2}\alpha y)\,\frac{1}{B}\int_0^1 Y\,dy$$

$$= y J_1(i^{3/2}\alpha y)\int_0^y Y\,dy\left\{-\int_0^1 X\,dy + \int_0^y X\,dy\right\}$$

$$- y J_1(i^{3/2}\alpha y)\,\frac{1}{B}\int_0^1 Y\,dy$$

$$= -y J_1(i^{3/2}\alpha y)\int_y^1 X\,dy \int_0^y Y\,dy \;- y J_1(i^{3/2}\alpha y)\,\frac{1}{B}\int_0^1 Y\,dy$$

$$= -y J_1(i^{3/2}\alpha y)\int_y^1 \frac{1}{y\left[J_1(i^{3/2}\alpha y)\right]^2}\,dy \int_0^y g(y) J_1(i^{3/2}\alpha y)\,dy$$

$$- \frac{y J_1(i^{3/2}\alpha y)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)\, J_1(i^{3/2}\alpha)}\int_0^1 g(y) J_1(i^{3/2}\alpha y)\,dy \tag{9-11}$$

At the boundary of the tube where $y = \frac{r}{R} = 1$, equation 9-11 reduces to

$$q(y)\Big]_{y=1} = -\frac{J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)\, J_1(i^{3/2}\alpha)} \int_0^1 g(y)\, J_1(i^{3/2}\alpha y)\, dy$$

$$= -\frac{1}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)} \int_0^1 g(y)\, J_1(i^{3/2}\alpha y)\, dy \qquad (9\text{-}12)$$

Equation 9-12 can be put into a more convenient form by using integration by parts according to

$$\int u\, dv = uv - \int v\, du$$

where $u = g(y)$

$$du = d[g(y)]$$

$$= d\left[\int_0^y f(y)(2y)\,dy\right]$$

$$= f(y)(2y)$$

and $g(y)\Big|_{y=0}^{y=1} = \int_0^1 f(y)(2y)\,dy$

Moreover, $dv = J_1(\alpha i^{3/2} y)$

$$v = \int J_1(\alpha i^{3/2} y)\,dy$$

$$= -\frac{1}{\alpha i^{3/2}} J_0(\alpha i^{3/2} y)$$

Thus, from equation 9-12 we have, upon integration by parts

$$q(y)\Big]_{y=1} = -\frac{1}{i^{3/2}\alpha J_0(i^{3/2}\alpha)}\left\{\left[g(y)\left(\frac{-1}{i^{3/2}\alpha}\right)J_0(i^{3/2}\alpha y)\right]_{y=1}^{y=1}\right.$$

$$\left. - \int_0^1 \left(\frac{-1}{i^{3/2}\alpha}\right) J_0(i^{3/2}\alpha y)\, f(y)\, 2y\, dy\right\}$$

$$= -\frac{1}{i^{3/2}\alpha J_0(i^{3/2}\alpha)}\left\{ g(y)\Big]_{y=0}^{y=1}\left(\frac{-1}{i^{3/2}\alpha}\right) J_0(i^{3/2}\alpha)\right.$$

$$\left. + \frac{1}{i^{3/2}\alpha}\int_0^1 J_0(i^{3/2}\alpha y)\, f(y)\, 2y\, dy\right\}$$

$$= \left(\frac{1}{i^3\alpha^2}\right) g(y)\Big]_{y=0}^{y=1} - \frac{1}{i^3\alpha^2 J_0(i^{3/2}\alpha)}\int_0^1 J_0(i^{3/2}\alpha y)\, f(y)\, 2y\, dy$$

$$= \left(\frac{1}{i^3\alpha^2}\right)\int_0^1 f(y)(2y)\, dy - \frac{1}{i^3\alpha^2}\int_0^1 \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\, f(y)(2y)\, dy$$

$$= \left(\frac{1}{i^3\alpha^2}\right)\int_0^1 \left\{1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right\} f(y)(2y)\, dy \qquad (9\text{-}13)$$

THE HARMONIC COMPONENTS OF THE LONGITUDINAL FLUID VELOCITY

In equation 9-1 for the longitudinal fluid velocity, we use the following harmonic representations for $\partial p/\partial z$, w and u:

$$-\frac{\partial p}{\partial z} = A_0 + A_1 e^{in\left(t - \frac{z}{c_1}\right)} + A_1^* e^{-in\left(t - \frac{z}{c_1^*}\right)} + \cdots$$

$$+ A_m e^{imn\left(t - z/c_m\right)} + A_m^* e^{-imn\left(t - z/c_m^*\right)} + \cdots$$

$$w = w_0 + w_1 e^{in\left(t - z/c_1\right)} + w_1^* e^{-in\left(t - z/c_1^*\right)} + \cdots$$

$$+ w_m e^{imn\left(t - z/c_m\right)} + w_m^* e^{-imn\left(t - z/c_m^*\right)} + \cdots$$

$$u = u_0 + u_1 e^{in\left(t - z/c_1\right)} + u_1^* e^{-in\left(t - z/c_1^*\right)} + \cdots$$

$$+ u_m e^{imn\left(t - z/c_m\right)} + u_m^* e^{-imn\left(t - z/c_m^*\right)} + \cdots$$

In the above representation, $u_o = 0$, i.e., the radial velocity component has no steady component and is entirely periodic. w_o is the steady component of the longitudinal fluid velocity. w_1, w_1^*, ... are functions of y and not of z and t. c_m is the complex wave velocity of the m^{th} harmonic. We also note that the steady component of the flow just increases or decreases the amplitude of the flow and does not affect the frequency.

After substitution into equation 9-1 and collecting powers of e^{int}, we obtain the equations for the amplitudes of the harmonic components of the longitudinal fluid velocity, w. The details are as follows:

$$w = w_o + w_1 e^{in(t - z/c_1)} + w_1^* e^{-in(t - z/c_1^*)} + \cdots$$

$$+ w_m e^{imn(t - z/c_m)} + w_m^* e^{imn(t - z/c_m^*)} + \cdots$$

$$\frac{1}{y}\frac{\partial w}{\partial y} = \frac{1}{y}\left\{ \frac{dw_o}{dy} + \frac{dw_1}{dy} e^{in(t - z/c_1)} + \frac{dw_1}{dy} e^{-in(t - z/c_1^*)} + \cdots \right.$$

$$+ \frac{dw_m}{dy} e^{imn(t - z/c_m)} + \frac{dw_m^*}{dy} e^{-imn(t - z/c_m^*)} + \cdots$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{d^2 w_0}{dy^2} + \frac{d^2 w_1}{dy^2} e^{in(t - z/c_1)} + \frac{d^2 w_1^*}{dy^2} e^{-in(t - z/c_1^*)} + \cdots$$

$$+ \frac{d^2 w_m}{dy^2} e^{imn(t - z/c_m)} + \frac{d^2 w_m^*}{dy^2} e^{-imn(t - z/c_m^*)} + \cdots$$

$$\left(\frac{R^2}{\nu}\right)\frac{\partial w}{\partial t} = \frac{R^2}{\nu}\left\{ (w_1)(in) e^{in(t - z/c_1)} + (w_1^*)(-in) e^{-in(t - z/c_1^*)} + \cdots\right.$$

$$\left. + w_m(imn) e^{imn(t - z/c_m)} + w_m^*(-imn) e^{-imn(t - z/c_m^*)} + \cdots \right\}$$

$$\left(\frac{R^2}{\mu}\right)\frac{\partial p}{\partial z} = -\frac{R^2}{\mu}\left\{ A_0 + A_1 e^{in(t - z/c_1)} + A_1^* e^{-in(t - z/c_1^*)} + \cdots\right.$$

$$\left. + A_m e^{imn(t - z/c_m)} + A_m^* e^{-imn(t - z/c_m^*)} + \cdots \right\}$$

$$\left(\frac{R}{\nu}\right)(u)\frac{\partial w}{\partial y} = \frac{R}{\nu}\left\{\left[u_0 + u_1 e^{in(t - z/c_1)} + u_1^* e^{-in(t - z/c_1^*)} + \cdots\right.\right.$$

$$\left. + u_m e^{imn(t - z/c_m)} + u_m^* e^{-imn(t - z/c_m^*)} + \cdots \right]\left[\frac{dw_0}{dy} + \right.$$

$$\left.\left. + \frac{dw_1}{dy} e^{in(t - z/c_1)} + \frac{dw_1^*}{dy} e^{-in(t - z/c_1^*)} + \cdots \right]\right\}$$

Collecting corresponding powers of e^{int}, we obtain the equation for w_o:

$$\frac{d^2 w_o}{dy^2} + \frac{1}{y}\frac{dw_o}{dy} = -\frac{A_o R^2}{\mu} + \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_1 \frac{dw_1^*}{dy} + \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_1^* \frac{dw_1}{dy}$$

$$+ \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_2 \frac{dw_2^*}{dy} + \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_2^* \frac{dw_2}{dy}$$

$$+ \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_m \frac{dw_m^*}{dy} + \frac{1}{c_o}\cdot\frac{R^2}{\nu}\cdot\frac{c_o}{R}\cdot u_m^* \frac{dw_m}{dy} + \cdots$$

$$= -\frac{A_o R^2}{\mu} + \frac{1}{c_o}\cdot\frac{R^2}{\nu}\sum_m\left[\frac{c_o u_m}{R}\frac{dw_m^*}{dy} + \frac{c_o u_m^*}{R}\frac{dw_m}{dy}\right] \tag{9-14}$$

Similarly, the equation for w_1 is:

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = -\frac{A_1 R^2}{\mu}$$

$$+ \frac{1}{c_o}\frac{R^2}{\nu}\sum_m\left\{\frac{c_o u_m}{R}\frac{dw_{m-1}^*}{dy} + \frac{c_o u_{m-1}^*}{R}\frac{dw_m}{dy}\right\}$$

$$- \frac{i\alpha^2}{c_o}\sum_m\left\{\frac{m c_o}{c_m} - \frac{(m-1)c_o}{c_w^*}\right\} w_m w_{m-1}^* \tag{9-15}$$

Similarly for the other harmonics.

As with the radial expansion correction (see section VIII), if we substitute the known forms for u_m and w_m on the right-hand sides of equations 9-14, 9-15 and other such equations for the second, third, etc. harmonics, they become linear equations and accordingly, the effects of the interactions between the harmonics can be treated separately. In the next section, we shall write down the equations describing these individual interactions between the harmonics.

There are four different forms of interactions to be considered. These are:

1. The effect of the m^{th} harmonic on the steady flow, denoted by W(m,-m).
2. The effect on the m^{th} harmonic of its own interaction with the steady flow, W(m,0).
3. The effect on the $(k-m)^{th}$ harmonic of the interaction between the k^{th} and m^{th} harmonics, W(k,-m).
4. The effect on the $(m-k)^{th}$ harmonic of the interaction between the m^{th} and k^{th} harmonics, W(m,-k).

Note that W(k,-m) and W(m,-k) are symmetric.

THE EFFECT OF THE m^{th} HARMONIC ON THE STEADY FLOW

The equation describing this form of interaction is obtained from equation 9-14. Note that we may write

$$\left(\frac{1}{y}\right)\frac{d}{dy}\left(y\,\frac{dw}{dy}\right) = \frac{1}{y}\left\{\frac{dw}{dy} + y\,\frac{d^2w}{dy^2}\right\} = \left(\frac{1}{y}\right)\frac{dw}{dy} + \frac{d^2w}{dy^2}$$

The right-hand side of equation 9-14 consists of the sum:

(Poiseuille flow $\frac{A_0 R^2}{\mu}$) + (sum of all harmonic terms).

To take into account the effect of the m^{th} harmonic only, we take only the m^{th} terms on the right-hand side of equation 9-14. Thus the equation describing this form of interaction is

$$\left(\frac{1}{y}\right)\frac{d}{dy}\left(y\,\frac{dw}{dy}\right) = \left(\frac{1}{C_0}\right)\frac{R^2}{\nu}\left\{\frac{C_0 u_m}{R}\,\frac{dw_m^*}{dy} + \frac{C_0 u_m^*}{R}\,\frac{dw_m}{dy}\right\} \tag{9-16}$$

The form of the quantities $\frac{u_m}{R}$, $\frac{u_m^*}{R}$, $\frac{dw_m}{dy}$ and $\frac{dw_m^*}{dy}$ on the right-hand side of equation 9-16 are obtained as follows.

$$u_1 = \frac{inR}{2C}\left[C_1\,\frac{2J_1\left(i^{3/2}\alpha y\right)}{i^{3/2}\alpha\,J_0\left(i^{3/2}\alpha\right)} + \left(\frac{A_1}{\rho_0 C}\right)y\right] \tag{3-21}$$

$$\frac{u_1}{R} = \frac{in}{2C}\left[C_1\,\frac{2J_1\left(i^{3/2}\alpha y\right)}{i^{3/2}\alpha\,J_0\left(i^{3/2}\alpha\right)} + \left(\frac{A_1}{\rho_0 C}\right)y\right]$$

$$\frac{u_m}{R} = \frac{inm}{2c_m}\left[C_1 \, \frac{2J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_0\left(i^{3/2}\alpha_m\right)} + \frac{A_m}{\rho_0 c_m} y \right]$$

$$= \left(\frac{A_m}{\rho_0 c_m}\right)\frac{inm}{2c_m}\left[\left(\frac{\rho_0 c_m}{A_m}\right) C_1 \, \frac{2J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_0\left(i^{3/2}\alpha_m\right)} + y \right]$$

$$= \left(\frac{A_m}{imn\rho}\right)\frac{inm}{2c_m}\left[\eta_m \frac{2J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_0\left(i^{3/2}\alpha_m\right)} + y \right]$$

$$\frac{u_m^*}{R} = \left(\frac{A_m^*}{-imn\rho}\right)\left(\frac{-inm}{2c_m^*}\right)\left[\eta_m^* \frac{2J_1\left(i^{-3/2}\alpha_m y\right)}{\alpha_m i^{-3/2} J_0\left(i^{-3/2}\alpha_m\right)} + y \right]$$

$$w_1 = \frac{A_1}{\rho_0 c}\left[1 + \eta \, \frac{J_0\left(i^{3/2}\alpha y\right)}{J_0\left(i^{3/2}\alpha\right)} \right] \qquad (6\text{-}3)$$

$$w_m = \frac{A_m}{\rho_0 c_m}\left[1 + \eta_m \frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right]$$

$$\frac{dw_m}{dy} = \frac{A_m}{imn\rho}\left[-\eta_m \alpha_m i^{3/2}\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right]$$

$$\frac{dw_m}{dy} = \frac{A_m^*}{-imn\rho}\left[-\eta_m^* \alpha_m i^{-3/2}\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right]$$

Consider the first term on the right-hand side of equation 9-16:

$$\left(\frac{1}{c_o}\right)\frac{R^2}{\nu}\left\{c_o\left(\frac{u_m}{R}\right)\left(\frac{dw_{3m}^*}{dy}\right)\right\}$$

$$= \left(\frac{1}{c_o}\right)\frac{R^2}{\nu}\, c_o\left\{\frac{A_m}{imn\rho}\left(\frac{imn}{2c_m}\right)\left[y + \eta_m \frac{2\, J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_o\left(i^{3/2}\alpha_m\right)}\right]\cdot\right.$$

$$\left.\frac{A_m^*}{-imn\rho}\left[-\eta_m^*\alpha_m i^{-3/2}\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right]\right\}$$

$$= -\frac{i\alpha_m^2}{c_o}\left(\frac{c_o}{c_m}\right)\frac{A_m A_m^*}{m^2 n^2 \rho^2}\left[\frac{y}{2} + \eta_m \frac{J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_o\left(i^{3/2}\alpha_m\right)}\right]\cdot$$

$$\left[\eta_m^*\alpha_m i^{-3/2}\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right]$$

Consider the second term on the right-hand side of equation 9-16:

$$\left(\frac{1}{C_o}\right)\frac{R^2}{\nu}\left\{c_o\left(\frac{u_m^*}{R}\right)\frac{dw_m}{dy}\right\}$$

$$= \left(\frac{1}{C_o}\right)\frac{R^2 c_o}{\nu}\left\{\left(\frac{A_m^*}{-imn\rho}\right)\left(\frac{-imn}{2C_m^*}\right)\left[y+\eta_m^*\frac{2J_1(i^{-3/2}\alpha_m y)}{i^{-3/2}\alpha_m J_o(i^{-3/2}\alpha_m)}\right]\cdot\right.$$

$$\left.\left(\frac{A_m}{imn\rho}\right)\left[-\eta_m\alpha_m i^{3/2}\frac{J_1(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\right]\right\}$$

$$= \left(\frac{i\alpha_m^2}{C_o}\right)\left(\frac{A_m A_m^*}{m^2n^2\rho^2}\right)\left(\frac{c_o}{C_m^*}\right)\left[\frac{y}{2}+\eta_m^*\frac{J_1(i^{-3/2}\alpha_m y)}{i^{-3/2}\alpha_m J_o(i^{-3/2}\alpha_m)}\right]\cdot$$

$$\left[\eta_m\alpha_m i^{3/2}\frac{J_1(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\right]$$

Thus, from equation 9-16 we have:

$$\left(\frac{1}{y}\right)\frac{d}{dy}\left(y\frac{dw}{dy}\right) = \left(\frac{i\alpha_m^2}{c_o}\right)\left(\frac{A_m A_m^*}{m^2 n^2 \rho^2}\right)\left(\frac{c_o}{c_m^*}\right).$$

$$\left[\frac{y}{2} + \eta_m^* \frac{J_1\left(i^{-3/2}\alpha_m y\right)}{i^{-3/2}\alpha_m J_0\left(i^{-3/2}\alpha_m\right)}\right]\left[\eta_m \alpha_m i^{3/2}\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right]$$

$$-\left(\frac{i\alpha_m^2}{c_o}\right)\left(\frac{A_m A_m^*}{m^2 n^2 \rho^2}\right)\left(\frac{c_o}{c_m}\right)\left[\frac{y}{2} + \eta_m \frac{J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_0\left(i^{3/2}\alpha_m\right)}\right].$$

$$\left[\eta_m^* \alpha_m i^{-3/2}\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right]$$

$$= \frac{1}{c_o}\left(\frac{A_m A_m^*}{m^2 n^2 \rho^2}\right)\left\{\frac{i\alpha_m^2}{2}\left(\frac{c_o}{c_m^*}\right)\left[\eta_m \alpha_m i^{3/2}(y)\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right]\right.$$

$$-\frac{i\alpha_m^2}{2}\left(\frac{c_o}{c_m}\right)\left[\eta_m^* \alpha_m i^{-3/2}(y)\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right]$$

$$\left. + 2\alpha_m^2\left|\eta_m\right|^2\left(\frac{c_o}{c_m}\right)_{REAL}\cdot\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\cdot\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right\}$$

$$\left(\frac{1}{y}\right)\frac{d}{dy}\left(y\,\frac{dw}{dy}\right) = \frac{1}{c_0}\left(\frac{M_m}{mn\rho}\right)^2 \left\{ \frac{i\,\alpha_m^2}{2}\left(\frac{c_0}{c_m^*}\right)\cdot\right.$$

$$\left[\eta_m \alpha_m i^{3/2}\, y\,\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right] - \frac{i\,\alpha_m^2}{2}\left(\frac{c_0}{c_m}\right)\left[\eta_m^* \alpha_m i^{-3/2}\, y\,\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right]$$

$$\left. + \,2\alpha_m^2 \left|\eta_m\right|^2 \left(\frac{c_0}{c_m}\right)_{\text{REAL}} \frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)} \cdot \frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)} \right\}$$

In analogy with equation 8-23, we define the standard correction function as

$$\frac{1}{c_0}\left(\frac{M_m}{mn\rho}\right)^2 W\left(m,-m\right)$$

It follows that $$W(m,-m) = \frac{1}{4}\int_0^1 w(2y)\,dy \qquad (9\text{-}17)$$

where w is the solution of the equation

$$\left(\frac{1}{y}\right)\frac{d}{dy}\left(y\frac{dw}{dy}\right) = \frac{i\alpha_m^2}{2}\left(\frac{c_o}{c_m{}^*}\right)\left[\eta_m \alpha_m i^{3/2} y \frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right]$$

$$-\frac{i\alpha_m^2}{2}\left(\frac{c_o}{c_m}\right)\left[\eta_m^* \alpha_m i^{-3/2} y \frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right]$$

$$+2\alpha_m^2\left|\eta_m\right|^2\left(\frac{c_o}{c_m}\right)_{REAL}\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\cdot\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)} \qquad (9\text{-}18)$$

Now, we multiply both sides of equation 9-18 by y and integrate by parts to obtain the velocity w. We find that:

$$\frac{d}{dy}\left(y\frac{dw}{dy}\right) = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m{}^*}\right)\eta_m \alpha_m i^{3/2}\left[y^2\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right]$$

$$-\left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m}\right)\eta_m^*\alpha_m i^{-3/2}\left[y^2\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right]$$

$$+2\alpha_m^2\left|\eta_m\right|^2\left(\frac{c_o}{c_m}\right)_{REAL} y\,\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\cdot\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}$$

$$y\frac{dw}{dy} = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_0}{c_m^*}\right)\eta_m\, y^2\,\frac{J_2\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}$$

$$-\left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_0}{c_m}\right)\eta_m^*\, y^2\,\frac{J_2\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}$$

$$+\ 2\alpha_m^2\left(\frac{c_0}{c_m}\right)_{REAL}\left|\eta_m\right|^2\left(\frac{y}{-2i\alpha_m^2}\right)\Bigg\{\frac{i\,J_0\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\cdot\frac{d}{dy}\left[\frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(\alpha_m i^{3/2}\right)}\right]$$

$$+\ i\,\frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\cdot\frac{d}{dy}\left[\frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right] \qquad (9\text{-}19)$$

We now proceed as follows:

1) divide equation 9-19 through by y;

2) use the recurrence relation

$$y\,\frac{J_2(ky)}{J_0(k)} = \frac{2}{k}\,\frac{J_1(ky)}{J_0(k)} - y\,\frac{J_0(ky)}{J_0(k)}$$

3) integrate to obtain the expression for w.

Dividing 9-19 through by y we obtain:

$$\frac{dw}{dy} = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m^*}\right)\eta_m\, y\, \frac{J_2(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}$$

$$-\left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m}\right)\eta_m^*\, y\, \frac{J_2(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}$$

$$+\ 2\alpha_m^2\left(\frac{c_o}{c_m}\right)_{REAL}|\eta_m|^2\left(\frac{1}{-2i\alpha_m^2}\right)\left\{i\,\frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}\cdot\frac{d}{dy}\left[\frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\right]\right.$$

$$\left.+\ i\,\frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\cdot\frac{d}{dy}\left[\frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}\right]\right\}$$

Using the recurrence relation, we obtain

$$\frac{dw}{dy} = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m^*}\right)\eta_m\left[\left(\frac{2}{\alpha_m i^{3/2}}\right)\frac{J_1(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)} - y\,\frac{J_o(i^{3/2}\alpha_m y)}{J_o(i^{3/2}\alpha_m)}\right]$$

$$-\left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m}\right)\eta_m^*\left[\left(\frac{2}{\alpha_m i^{-3/2}}\right)\frac{J_1(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)} - y\,\frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}\right]$$

$$+ 2\alpha_m^2 \left(\frac{c_o}{c_m}\right)_{REAL} \left|\eta_m\right|^2 \left(\frac{1}{-2i\alpha_m^2}\right) \left\{ i\, \frac{J_o\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)} \cdot \frac{d}{dy}\left[\frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right]\right.$$

$$\left. + i\, \frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)} \cdot \frac{d}{dy}\left[\frac{J_o\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right]\right\}$$

Integrating with respect to y, we obtain:

$$w = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m^*}\right)\eta_m \left(\frac{2}{\alpha_m i^{3/2}}\right)\left\{\int \frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\,dy - \int y\,\frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\,dy\right\}$$

+ conjugate of above

$$+ 2\alpha_m^2 \left(\frac{c_o}{c_m}\right)_{REAL} \left|\eta_m\right|^2 \left(\frac{1}{-2i\alpha_m^2}\right)\left\{ \qquad\qquad\qquad\qquad \right\}$$

We may write the above equation as

$$w = \left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m^*}\right)\eta_m\left(\frac{2}{i^{3/2}\alpha_m^2}\right)\left\{1 - \frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right\}$$

$$+\left(\frac{i\alpha_m^2}{2}\right)\left(\frac{c_o}{c_m^*}\right)\eta_m\left\{\left(\frac{1}{\alpha_m i^{3/2}}\right)\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)} - \left(\frac{y}{\alpha_m i^{3/2}}\right)\frac{J_1\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\right\}$$

+ conjugate of these two terms

$$+\left(\frac{c_o}{c_m}\right)_{REAL}\left|\eta_m\right|^2\left\{1 - \frac{J_o\left(i^{3/2}\alpha_m y\right)}{J_o\left(i^{3/2}\alpha_m\right)}\ \frac{J_o\left(i^{-3/2}\alpha_m y\right)}{J_o\left(i^{-3/2}\alpha_m\right)}\right\}$$

(9-20)

In order to obtain the average velocity across the cross section of the tube, we integrate the expression for w in equation 9-20 with respect to y from y=0 to y=1 and obtain:

$$W(m,-m) = \frac{1}{4}\int_0^1 w(2y)\ dy$$

Thus,

$$W(m,-m) = \frac{1}{4}(-\eta)\left(\frac{c_c}{c^*}\right)M_{10}' \, e^{i\epsilon_{10}'}$$

$$+ \left(\frac{i\alpha^2}{8}\right)\left(\frac{c_o}{c^*}\right)\eta\left(\frac{1}{i^{3/2}\alpha}\right)\frac{J_1\left(i^{3/2}\alpha\right)}{J_o\left(i^{3/2}\alpha\right)}$$

$$- \left(\frac{i\alpha^2}{4}\right)\left(\frac{c_o}{c^*}\right)\eta\left(\frac{1}{i^3\alpha^2}\right)\frac{J_2\left(i^{3/2}\alpha\right)}{J_o\left(i^{3/2}\alpha\right)}$$

together with its conjugate

$$+ \frac{1}{4}\left(\frac{c_c}{c}\right)_{REAL}\left|\eta^2\right|\left\{1 - \frac{2\,J_1\left(i^{3/2}\alpha\right)}{i^{3/2}\alpha\,J_o\left(i^{3/2}\alpha\right)}\right\}_{REAL} \tag{9-21}$$

From the relation

$$2nJ_n(x) = xJ_{n-1}(x) + J_{n+1}(x)$$

for n=1, we may write

$$\frac{J_2(x)}{J_0(x)} = \left(\frac{2}{x}\right)\frac{J_1(x)}{J_0(x)} - 1$$

or

$$-\frac{J_2(x)}{J_0(x)} = 1 - \left(\frac{2}{x}\right)\frac{J_1(x)}{J_0(x)}$$

or

$$\left(\frac{1}{i^2}\right)\frac{J_2(x)}{J_0(x)} = 1 - \left(\frac{2}{x}\right)\frac{J_1(x)}{J_0(x)}$$

or

$$\left(\frac{1}{i^2}\right)\frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)} = 1 - \left(\frac{2}{i^{3/2}\alpha}\right)\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}$$

$$= M'_{10}(\alpha)\, e^{i\,\varepsilon'_{10}(\alpha)}$$

In equation 9-21, the sum of the first and third terms may be written as

$$\left(\frac{1}{4}\right)(-\eta)\left(\frac{c_o}{c^*}\right)M'_{10}e^{i\epsilon'_{10}} + \left(\frac{i\alpha^2}{4}\right)\left(\frac{c_o}{c^*}\right)(-\eta)\left(\frac{1}{i^3\alpha^2}\right)\frac{J_2(i^{3/2}\alpha)}{J_o(i^{3/2}\alpha)}$$

$$= \left(\frac{1}{4}\right)(-\eta)\left(\frac{c_o}{c^*}\right)M'_{10}e^{i\epsilon'_{10}} + \left(\frac{1}{4}\right)(-\eta)\left(\frac{c_o}{c^*}\right)M'_{10}e^{i\epsilon'_{10}}$$

$$= \left(\frac{1}{2}\right)(-\eta)\left(\frac{c_o}{c^*}\right)M'_{10}e^{i\epsilon'_{10}}$$

and the conjugate of this result is

$$\frac{1}{2}(-\eta^*)\left(\frac{c_o}{c^*}\right)M'_{10}e^{-i\epsilon'_{10}}$$

The second term in equation 9-21 may be written as

$$\left(\frac{i\alpha^2}{8}\right)\left(\frac{c_o}{c^*}\right)\eta\left(\frac{1}{i^{3/2}\alpha}\right)\frac{J_1(i^{3/2}\alpha)}{J_o(i^{3/2}\alpha)}$$

$$= \left(\frac{i\alpha^2}{8}\right)\left(\frac{c_o}{c^*}\right)\eta\left(\frac{1}{2}\right)\left(1 - M'_{10}e^{i\epsilon'_{10}}\right)$$

and the conjugate of this expression is

$$\left(-\frac{i\alpha^2}{8}\right)\left(\frac{c_0}{c}\right)\left(\eta^*\right)\left(\frac{1}{2}\right)\left(1 - M_{10}' e^{-i\varepsilon_{10}'}\right)$$

The last term in equation 9-21 may be written as

$$\left(\frac{1}{4}\right)\left(\frac{c_0}{c}\right)_{REAL} \left|\eta^2\right| \left\{ M_{10}' e^{i\varepsilon_{10}'} \right\}_{REAL}$$

$$= \left(\frac{1}{4}\right)\left(\frac{c_0}{c}\right)_{REAL} \left|\eta^2\right| M_{10}' \cos \varepsilon_{10}'$$

Thus, equation 9-21 may be written in the form

$$W(m,-m) = \frac{1}{2}(-\eta)\left(\frac{c_0}{c^*}\right) M_{10}' e^{i\varepsilon_{10}'} + \frac{1}{2}(-\eta^*)\left(\frac{c_0}{c}\right) M_{10}' e^{-i\varepsilon_{10}'}$$

$$+ \frac{1}{4}\left|\eta^2\right|\left(\frac{c_0}{c}\right)_{REAL} M_{10}' \cos \varepsilon_{10}'$$

$$+ \frac{i\alpha^2}{8}(\eta)\left(\frac{c_0}{c^*}\right)\frac{1}{2}\left(1 - M_{10}' e^{i\varepsilon_{10}'}\right)$$

$$- \frac{i\alpha^2}{8}(\eta^*)\left(\frac{c_0}{c}\right)\frac{1}{2}\left(1 - M_{10}' e^{-i\varepsilon_{10}'}\right) \tag{9-22}$$

In the limiting condition of very stiff constraint, equation 9-22 reduces to the form

$$W(m,-m) = \frac{\sqrt{3}}{2}\left(M_{10}'\right)^{1/2}\cos\frac{3\varepsilon_{10}'}{2} + \frac{\sqrt{3}}{8}\left(M_{10}'\right)^{1/2}\cos\frac{\varepsilon_{10}'}{2}\cos\varepsilon_{10}'$$

$$+\left(\frac{\alpha^2}{8M_{10}'}\right)\frac{\sqrt{3}}{2}\left\{\left(M_{10}'\right)^{1/2}\sin\frac{\varepsilon_{10}'}{2} - \left(M_{10}'\right)^{3/2}\sin\frac{3\varepsilon_{10}'}{2}\right\}$$

(9-23)

This correction, W(m,-m), for the quadratic terms in the Navier-Stokes equation is in the same direction as the steady stream. The combined effect of this correction, W(m,-m), and the correction due to finite expansion, E(m,-m), may be written as

$$T(m,-m) = W(m,-m) + E(m,-m)$$

$$= \frac{\sqrt{3}}{2}\left(M_{10}'\right)^{1/2}\left[\cos\frac{3\varepsilon_{10}'}{2} + \frac{1}{4}\cos\varepsilon_{10}'\cos\frac{\varepsilon_{10}'}{2} + \frac{1}{2}M_{10}'\cos\frac{\varepsilon_{10}'}{2}\right.$$

$$\left.+\frac{\alpha^2}{8M_{10}'}\left(\sin\frac{\varepsilon_{10}'}{2} - M_{10}'\sin\frac{3\varepsilon_{10}'}{2}\right)\right]$$

The variation of the correction T(m,-m) as a function of α is shown in figure 54. From this figure, note that the steady flow is augmented for all values of α less than 10, but that as α increases further, the effect of inertia is dominant and the steady flow is hindered by the presence of the oscillatory terms. The amount of this combined correction, T(m,-m), for the results of McDonald's work is given in table VIII, where c_0 is taken to be 10 meters/sec. In McDonald's experiment, the measured steady term was 15 cm/sec. Thus, this correction is about 12%, and by no means negligible.

TABLE VIII

The Combined Correction, T(m,-m), for $\bar{w}_0$. c_0 = 10 Meters/Sec

m	$\frac{1}{c_0}\left(\frac{M}{mn\rho}\right)^2$	T	Contribution to $\bar{w}_0$
1	2.696	0.537	1.448
2	0.488	0.539	0.263
3	0.031	0.455	0.014
4	0.00003	---	---

Total: 1.725 cm/sec.

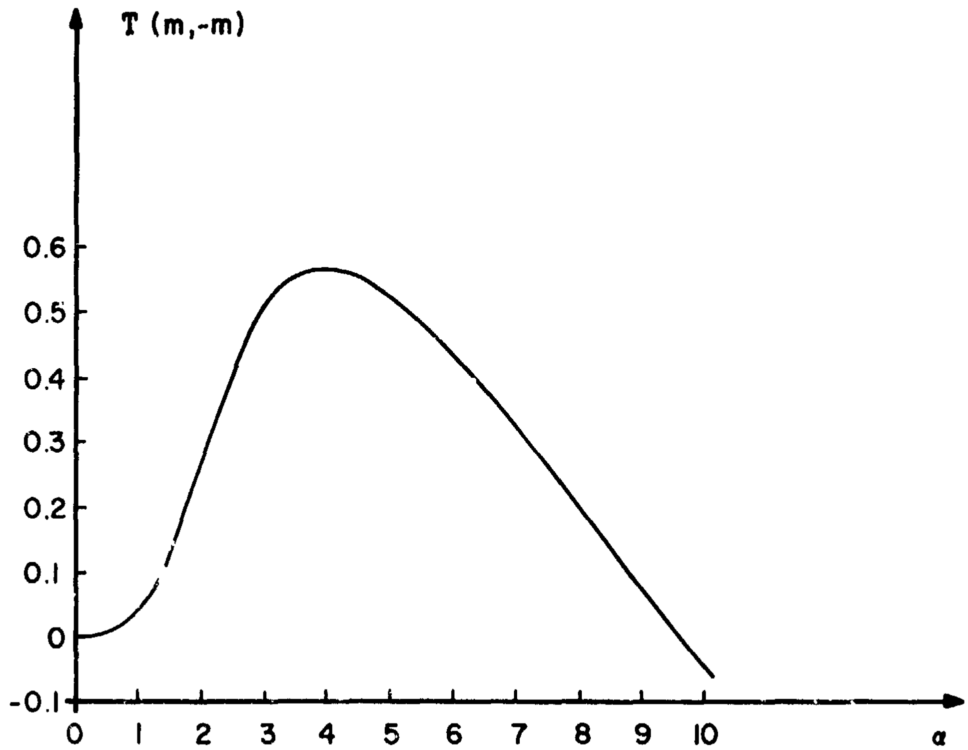

Figure 54. Variation of the Combined Correction Function to Steady Flow, T(m,-m), with Respect to α.

THE EFFECT ON THE m^{th} HARMONIC OF ITS OWN INTERACTION WITH THE STEADY FLOW

The equation describing the effect on the m^{th} harmonic of its own interaction with the steady flow may be obtained by referring to equation 9-15 and considering only the m^{th} term. This equation is

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3 \alpha_m^2 w = \frac{R^2}{c_o \nu}\left(\frac{c_o U_m}{R}\right)\frac{dw_o}{dy} - \frac{i\alpha_m^2}{c_o}\left(\frac{c_o}{c_m}\right) w_o w_m$$

(9-24)

In analogy with the preceding subsection, the equation for the standard correction function is

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3 \alpha_m^2 w = -\alpha_m^2\left(\frac{c_o}{c_m}\right)\left[y^2 + \eta_m \frac{(2y) J_1\left(i^{3/2}\alpha_m y\right)}{i^{3/2}\alpha_m J_0\left(i^{3/2}\alpha_m\right)}\right]$$

$$-\alpha_m^2\left(\frac{c_o}{c_m}\right)\left[1 + \eta_m \frac{J_0\left(i^{3/2}\alpha_m y\right)}{J_0\left(i^{3/2}\alpha_m\right)}\right]\left(1 - y^2\right)$$

(9-25)

Equation 9-25 is analogous to equation 9-3, where f(y) is the right-hand side of equation 9-25. Solving equation 9-25, we obtain the standard correction function in the form

$$W(m,0) = -i\left(\frac{c_o}{c_m}\right)\int_0^1 \left[1 - \frac{J_0(i^{3/2}\alpha_m y)}{J_0(i^{3/2}\alpha_m)}\right]\left[y^2 + \eta_m \frac{2y J_1(i^{3/2}\alpha_m y)}{i^{3/2}\alpha_m J_0(i^{3/2}\alpha_m)}\right] 2y\,dy$$

$$-i\left(\frac{c_o}{c_m}\right)\int_0^1 \left[1 - \frac{J_0(i^{3/2}\alpha_m y)}{J_0(i^{3/2}\alpha_m)}\right]\left[1 + \eta_m \frac{J_0(i^{3/2}\alpha_m y)}{J_0(i^{3/2}\alpha_m)}\right](1-y^2)\,2y\,dy$$

(9-26)

All the terms in these two integrals can be expressed in terms of $J_0(\alpha i^{3/2})$ and $J_1(\alpha i^{3/2})$. The necessary reduction formulae can be found in Watson: "Theory of Bessel Functions," Chapter V. It is, however, simpler and quicker to evaluate them by direct numerical integration.

We have seen earlier that when the pressure gradient is in real form, the radial expansion correction for the m^{th} harmonic has the form

$$E(m,0) = \frac{1}{c_o}\left(\frac{M_o}{mn\rho}\right)\left(\frac{M_m}{mn\rho}\right)|E(m,0)| \cos\left[mnt + \phi_m + \text{phase } E(m,0)\right]$$

Similarly, for the pressure gradient in real form, the expression for W(m,0), described by equation 9-26, must be combined with its conjugate. Thus, in analogy with the expression for E(m,0) above, the complete interaction term for the m^{th} harmonic is

$$W(m,0) = \frac{1}{c_0}\left(\frac{M_0 R^2}{4\mu}\right)\left(\frac{M_m}{mn\rho}\right)|W(m,0)| \cos\left[mnt + \phi_m + \text{phase } W(m,0)\right]$$

THE EFFECT ON THE (k-m)th HARMONIC OF THE INTERACTION BETWEEN THE kth AND mth HARMONICS

The equation describing the effect on the (k-m)th harmonic of the interaction between the kth and mth harmonics may be obtained by reference to equation 9-15. This equation is

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3 \alpha^2_{k-m} w = \frac{R^2}{c_o \nu}\left(\frac{c_o U_k}{R}\frac{dw^*_m}{dy} + \frac{c_o U^*_m}{R}\frac{dw_k}{d\alpha_0}\right)$$

$$- \frac{i\alpha^2}{c_o}\left(\frac{kc_o}{c_k} - \frac{mc_o}{c^*_m}\right) w_k w^*_m \qquad (9\text{-}27)$$

In analogy with the preceding subsection, the equation for the standard correction function is

$$\frac{d^2 w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + i^3 \alpha^2_{k-m} w$$

$$= i\alpha^2_k\left(\frac{c_o}{c_k}\right)\left\{\frac{y}{2} + \eta_k \frac{J_1(i^{3/2}\alpha_k y)}{i^{3/2}\alpha_k J_o(i^{3/2}\alpha_k)}\right\}(-\eta^*_k)(\alpha_m i^{-3/2})\frac{J_1(i\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}$$

$$- i\alpha^2_m\left(\frac{c_o}{c^*_m}\right)\left\{\frac{y}{2} + \eta^*_m \frac{J_1(i^{-3/2}\alpha_m y)}{i^{-3/2}\alpha_m J_o(i^{-3/2}\alpha_m)}\right\}(-\eta_k)(\alpha_k i^{3/2})\frac{J_1(i^{3/2}\alpha_k y)}{J_o(i^{-3/2}\alpha_k)}$$

$$- i\alpha^2\left(\frac{kc_o}{c_k} - \frac{mc_o}{c^*_m}\right)\left\{1 + \eta_k \frac{J_o(i^{3/2}\alpha_k y)}{J_o(i^{3/2}\alpha_k)}\right\}\left\{1 + \eta^*_m \frac{J_o(i^{-3/2}\alpha_m y)}{J_o(i^{-3/2}\alpha_m)}\right\} \qquad (9\text{-}28)$$

Again, the standard correction function, W(k,-m), is obtained, as earlier, in the form

$$W(k,-m) = \frac{1}{i^3 \alpha_{k-m}^2} \int_0^1 \left[1 - \frac{J_0(i^{3/2} \alpha_{k-m} y)}{J_0(i^{3/2} \alpha_{k-m})} \right] f(y)(2y)\, dy$$

where f(y) is given by the right-hand side of equation 9-28. Thus

$$W(k,-m) = \frac{1}{i^3 \alpha_{k-m}^2} \int_0^1 (i \alpha_k^2) \frac{C_0}{C_k} \left\{ \frac{y}{2} + \eta_k \frac{J_1(i^{3/2} \alpha_k y)}{i^{3/2} \alpha_k J_0(i^{3/2} \alpha_k)} \right\}$$

$$(-\eta_m^*)(\alpha_m i^{-1/2}) \frac{J_1(i^{-3/2} \alpha_m y)}{J_0(i^{-3/2} \alpha_m)} \left\{ 1 - \frac{J_0(i^{3/2} \alpha_{k-m} y)}{J_0(i^{3/2} \alpha_{k-m})} \right\} (2y)\, dy$$

$$+ \frac{1}{i^3 \alpha_{k-m}^2} \int_0^1 (-i \alpha_m^2) \frac{C_0}{C_m^*} \left\{ \frac{y}{2} + \eta_m^* \frac{J_1(i^{-3/2} \alpha_m y)}{i^{-3/2} \alpha_m J_0(i^{-3/2} \alpha_m)} \right\}$$

$$(-\eta_k)(\alpha_k i^{3/2}) \frac{J_1(i^{3/2} \alpha_k y)}{J_0(i^{3/2} \alpha_k)} \left\{ 1 - \frac{J_0(i^{3/2} \alpha_{k-m} y)}{J_0(i^{3/2} \alpha_{k-m})} \right\} (2y)\, dy$$

$$+ \frac{1}{i^3 \alpha_{k-m}^2} \int_0^1 (-i \alpha^2) \left\{ \frac{kC_0}{C_k} - \frac{mC_0}{C_m^*} \right\} \left\{ 1 + \eta_k \frac{J_0(i^{3/2} \alpha_k y)}{J_0(i^{3/2} \alpha_k)} \right\}$$

$$\left\{ 1 + \eta_m^* \frac{J_0(i^{-3/2} \alpha_m y)}{J_0(i^{-3/2} \alpha_m)} \right\} \left\{ 1 - \frac{J_0(i^{3/2} \alpha_{k-m} y)}{J_0(i^{3/2} \alpha_{k-m})} \right\} (2y)\, dy$$

In the first, second and third integrals above, we note that

$$\left(\frac{1}{i^3\alpha^2_{k-m}}\right)\left(i\alpha^2_k\right) = (-1)\left(k\alpha^2\right)\left[\frac{1}{(k-m)\alpha^2}\right] = -\left(\frac{k}{k-m}\right)$$

$$\left(\frac{1}{i^3\alpha^2_{k-m}}\right)\left(-i\alpha^2_m\right) = \frac{m\alpha^2}{(k-m)\alpha^2} = \frac{m}{k-m}$$

and $$\left(\frac{1}{i^3\alpha^2_{k-m}}\right)\left(-i\alpha^2\right) = \frac{1}{k-m}$$

Moreover, since $$\frac{c_0^2}{c^2} = \frac{1-\sigma^2}{1-F_0(\alpha)} = \left(\frac{3}{4}\right)\frac{1}{1-F_{10}(\alpha)} = \left(\frac{3}{4}\right)\frac{e^{-i\varepsilon(\alpha)}}{M'_{10}(\alpha)}$$

we write $$\frac{c_0^2}{c_k^2} = \left(\frac{3}{4}\right)\frac{e^{-i\varepsilon(\alpha_k)}}{M'_{10}(\alpha_k)}$$

$$\frac{c_0}{c_k} = \left[\left(\frac{3}{4}\right)\frac{e^{-i\varepsilon(\alpha_k)}}{M'_{10}(\alpha_k)}\right]^{1/2}$$

$$\frac{C_o}{C_m} = \left[\left(\frac{3}{4}\right) \frac{e^{-i\varepsilon(\alpha_m)}}{M_{10}'(\alpha_m)} \right]^{1/2}$$

$$\frac{C_o}{C_m^*} = \left[\left(\frac{3}{4}\right) \frac{e^{i\varepsilon(\alpha_m)}}{M_{10}'(\alpha_m)} \right]^{1/2}$$

Finally, we note that for the limiting condition of very stiff constraint, the expression for the correction, W(k,-m), has the form

$$W(k,-m) = -\left(\frac{k}{k-m}\right) \left[\left(\frac{3}{4}\right) \frac{e^{-i\varepsilon_{10}'(\alpha_k)}}{M_{10}'(\alpha_k)} \right]^{1/2}$$

$$\int_0^1 \left\{ \frac{y}{2} - \frac{J_1\left(i^{3/2}\alpha_k y\right)}{i^{3/2}\alpha_k J_0\left(i^{3/2}\alpha_k\right)} \right\} \left(\alpha_m i^{-3/2}\right) \frac{J_1\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}$$

$$\left\{ 1 - \frac{J_0\left(i^{3/2}\alpha_{k-m} y\right)}{J_0\left(i^{3/2}\alpha_{k-m}\right)} \right\} (2y)\, dy$$

$$+\left(\frac{m}{k-m}\right)\left[\left(\frac{3}{4}\right)\frac{e^{i\,\varepsilon_{10}'(\alpha_m)}}{M_{10}'(\alpha_m)}\right]^{1/2}\int_0^1\left\{\frac{y}{2}-\frac{J_1\left(i^{-3/2}\alpha_m y\right)}{i^{-3/2}\alpha_m\,J_0\left(i^{-3/2}\alpha_m\right)}\right\}$$

$$\left(\alpha_k i^{3/2}\right)\frac{J_1\left(i^{3/2}\alpha_k y\right)}{J_0\left(i^{3/2}\alpha_k\right)}\left\{1-\frac{J_0\left(i^{3/2}\alpha_{k-m} y\right)}{J_0\left(i^{3/2}\alpha_{k-m}\right)}\right\}(2y)\,dy$$

$$+\left(\frac{1}{k-m}\right)\left\{k\left[\left(\frac{3}{4}\right)\frac{e^{-i\,\varepsilon_{10}'(\alpha_k)}}{M_{10}'(\alpha_k)}\right]^{1/2}-m\left[\left(\frac{3}{4}\right)\frac{e^{i\,\varepsilon_{10}'(\alpha_m)}}{M_{10}'(\alpha_m)}\right]^{1/2}\right\}$$

$$\int_0^1\left[1-\frac{J_0\left(i^{3/2}\alpha_k y\right)}{J_0\left(i^{3/2}\alpha_k\right)}\right]\left[1-\frac{J_0\left(i^{-3/2}\alpha_m y\right)}{J_0\left(i^{-3/2}\alpha_m\right)}\right]$$

$$\left[1-\frac{J_0\left(i^{3/2}\alpha_{k-m} y\right)}{J_0\left(i^{3/2}\alpha_{k-m}\right)}\right](2y)\,dy$$

When the W(k,m) are known, the expressions for the corrected components of the average velocity can be written down, being similar in form to equations 8-34 - 8-37, with the W(k,m) taking the place of the E(k,m). The coefficients multiplying the W(k,m) in these expressions will be the same as those multiplying the E(k,m) in equations 8-34 - 8-37, except for the W(m,0) which will be as shown in equation 9-25. Except for the interaction with the steady flow, therefore, the E(k,m) and the W(k,m) can be combined into a single standard correction function, T(k,m). For convenience, tables of T(k,m) over a full range of values of α up to the fourth harmonic may be prepared.

In order to make an estimate of the magnitude of the correction for the same experimental results as in section VIII, the values of the W(k,m) for $\alpha = 3.34$, k and m ≤ 4, were calculated by numerical quadrature. The trapezoidal rule was used for integration, one hundred ordinates being taken in the range $0 \leq y \leq 1$. The values of the W(k,m) are given in table IX.

TABLE IX

Values of W(k,m) for $\alpha = 3.34$

k, m	W_{RE}	W_{im}	$\|W\|$	ph{W}
1, 0	0.6745	-0.4100	0.7893	-31.29°
2, 0	0.4986	-0.6424	0.8132	-52.13°
3, 0	0.3961	-0.7106	0.8135	-60.86°
4, 0	0.3369	-0.7423	0.8152	-65.59°
1, 1	0.0966	0.3030	0.3180	72.32°
2, 1	0.3130	0.6453	0.7172	64.12°
3, 1	0.3110	0.6575	0.7273	64.69°
2, -1	0.2495	0.9197	0.9529	74.82°
3, -1	0.2782	0.4694	0.5371	58.80°
4, -1	0.2870	0.2381	0.3729	39.68°
3, -2	0.1450	1.4288	1.4361	84.20°
4, -2	0.3032	0.9189	0.9676	71.74°
4, -3	0.0261	1.7606	1.7608	90.85°
2, 2	0.2392	0.2256	0.3288	43.32°

These values of the W(k,m) were substituted in the expressions for the velocity components (i.e., those corresponding to equations 8-34 through 8-37 above), together with the components of McDonald's observed pressure gradient. The resulting values of the coefficients in the Fourier series for the average velocity are given in table X, together with the values of the coefficients when this correction and that for finite expansion are combined.

TABLE X

Values of the Fourier Coefficients for the Calculation of the Average Velocity, with and without the Inertia Term Correction, and with the Combined Correction

Harmonic	Quadratic Term Correction Only. Coefficient of		Combined Correction Coefficient of	
	cos mnt	sin mnt	cos mnt	sin mnt
1	22.51	33.94	24.37	32.55
2	-31.56	13.92	-33.15	15.28
3	-7.35	-10.15	-6.72	-10.36
4	-0.94	-5.34	-0.36	-4.95

The average velocity, with the combined correction, is shown in figure 55. The full line shows the uncorrected average velocity, and the discrete points are the values of the corrected average velocity, plotted at intervals of 15°. The correction increases the predicted value of the systolic peak by about 5%, and, moreover, predicts greater backflow. The differences between the corrected and uncorrected values are small, never exceeding 7 cm/sec. Thus, since these corrections are exaggerated, the value of c_0 (= 500 cm/sec) taken being about two-thirds of its real value, these nonlinear corrections would seem to be an unnecessary refinement.

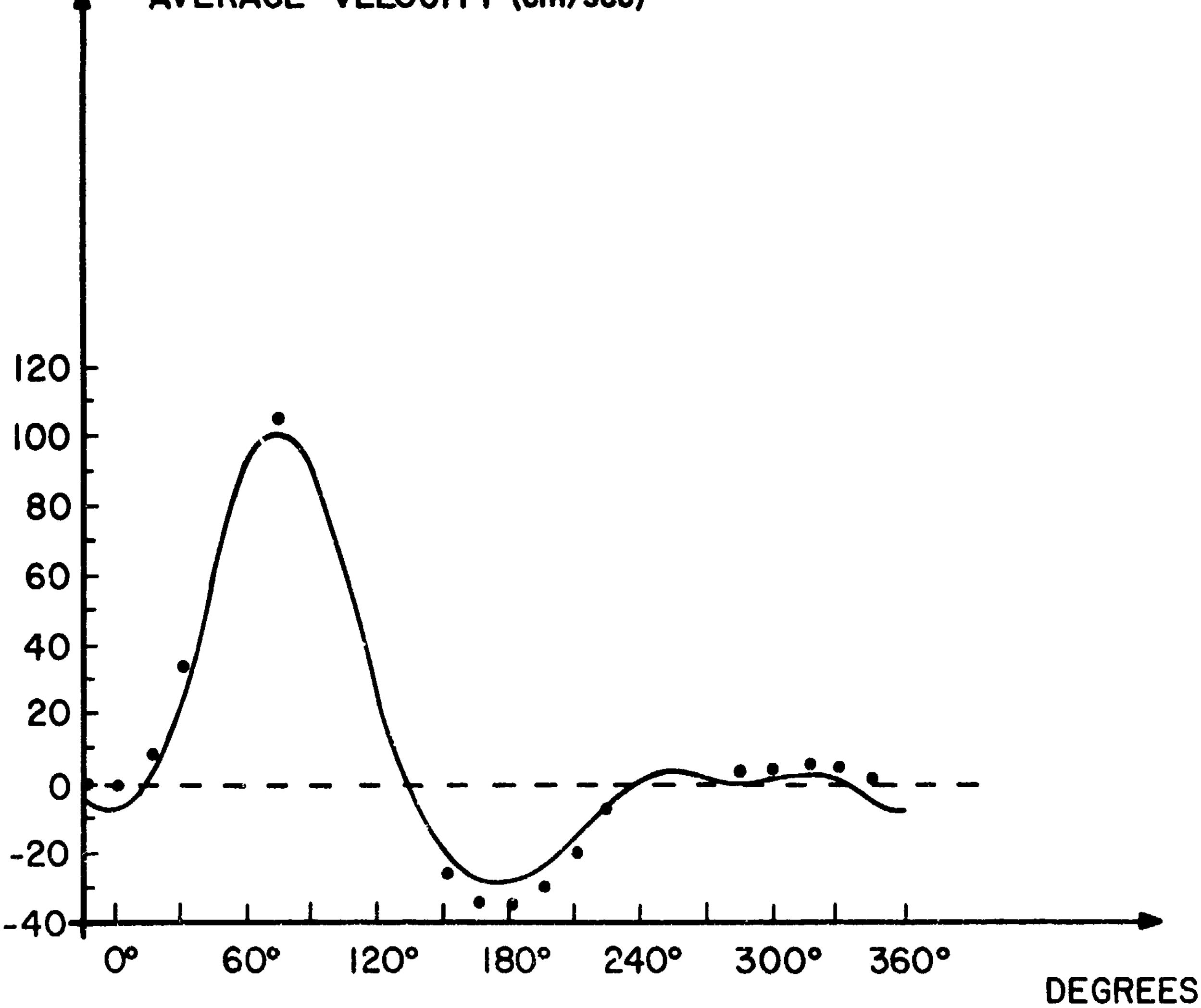

Figure 55. Variation in average velocity over one cycle in the femoral artery of the dog, calculated from the observed pressure gradient of figure 14.

Full line: Without nonlinear corrections.

Isolated points: With combined correction for c_0 = 500 cm/sec.

SECTION X

THE "EXACT" SOLUTION FOR OSCILLATORY MOTION IN THE PRESENCE OF A STEADY STREAM

INTRODUCTION

In the major arteries, the oscillatory components of the fluid velocity are at least as large as and very often considerably larger than the steady stream components. A solution for the fluid velocity, taking into account the interaction of these two factors and called the interaction velocity, will be obtained in terms of a confluent hypergeometric function neglecting the generation of higher harmonics. It is assumed that the higher harmonics can be accounted for by perturbation theory. Next, under the assumption that the velocity of the steady stream is small as compared with the pressure-wave velocity, an approximation to the above solution is obtained in terms of Bessel functions.

THE INTERACTION VELOCITY WHEN THE STEADY STREAM VELOCITY IS SMALL COMPARED WITH THE WAVE VELOCITY

We will assume that the pressure gradient, the longitudinal and radial components of the fluid velocity, may be represented respectively as follows:

$$-\frac{\partial p}{\partial z} = A_0 + A_1\, e^{in(t - z/c)} \tag{10-1}$$

$$w = w_0 + w_1\, e^{in(t - z/c)} \tag{10-2}$$

$$u = u_0 + u_1\, e^{in(t - z/c)} \tag{10-3}$$

Here, A_0, w_0 and u_0 are the values of the steady components of the pressure gradient, w and u respectively. A_1, w_1 and u_1 are the amplitudes of the oscillatory components of the pressure gradient, w and u respectively.

From the equation of continuity in the form

$$\frac{1}{y}\frac{d}{dy}(u_1 \cdot y) = \left(\frac{inR}{c}\right) w_1 \tag{3-41}$$

we note that

$$\frac{d}{dy}(u_1 \cdot y) = \left(\frac{inR}{c}\right) w_1 y$$

and on integration

$$u_1 y = \frac{inR}{2c} \int_0^y w_1(2y)\ dy$$

$$= (\frac{inR}{2c}) q_1$$

where q_1 is the stream function as defined in section IX.

From the earlier expression for the longitudinal component of the fluid velocity

$$w = w(y,t) = \frac{MR^2}{4\mu}(1-y^2)\cos(nt - \varphi) \tag{2-23}$$

we obtain an expression for the steady component, w_0,

$$w_0 = w_0(y) = \frac{A_0 R^2}{4\mu}(1 - y^2) \tag{10-4}$$

where we have used A_0 as the value of M for $n = 0$ and $\phi = 0$. At the center of the tube, $y = r/R = 0$ and the value of w_0 is

$$w_0 \Big|_{r=0} = \frac{A_0 R^2}{4\mu}$$

Since the average value of the steady component $\bar{w}_0 = (w_0/2) = (A_0 R^2/8\mu)$, we may write equation 10-4 in the form

$$w_0 = 2(\frac{A_0 R^2}{8\mu})(1 - y^2) = 2\bar{w}_0(1 - y^2) \tag{10-5}$$

Differentiating equation 10-5 with respect to y, we obtain the variation of the steady component of the longitudinal fluid velocity with respect to the radius of the tube

$$\frac{dw_0}{dy} = - 4\bar{w}_0 y$$

Finally, we shall take $\qquad u_0 = 0$

Substituting the values

$$-\frac{\partial p}{\partial z} = A_o + A_1 e^{in(t - z/c)} \tag{10-1}$$

$$w = w_o + w_1 e^{in(t - z/c)} = 2\bar{w}_o(1-y^2) + w_1 e^{in(t - z/c)}$$

$$u = u_o + u_1 e^{in(t - z/c)} \tag{10-3}$$

$$w_o = 2\bar{w}_o(1-y^2) \tag{10-5}$$

$$\frac{dw_o}{dy} = -4\bar{w}_o y$$

$$u_o = 0$$

into the equation for the longitudinal fluid velocity

$$\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{R^2}{\nu}\frac{\partial w}{\partial t} = \frac{R^2}{\mu}\frac{\partial p}{\partial z} + \left(\frac{R}{\nu}\right) u \frac{\partial w}{\partial y} + \left(\frac{R^2}{\nu}\right) w \frac{\partial w}{\partial z} \tag{9-1}$$

we obtain, after some computations, the equation

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 w_1 = -\frac{A_1 R^2}{\mu} - i\alpha^2\left(\frac{2\bar{w}_0}{c}\right) q_1 - i\alpha^2\left(\frac{2\bar{w}_0}{c}\right)(1-y^2) w_1$$

(10-6)

Now, we set $b^2 = (2\bar{w}_0/c)$, and note that the third term on the right-hand side may be written as

$$-i\alpha^2\left(\frac{2\bar{w}_0}{c}\right)(1-y^2) w_1 = -i\alpha^2 b^2 (1-y^2) w_1$$

The second term on the right-hand side has the form

$$-i\alpha^2\left(\frac{2\bar{w}_0}{c}\right) q_1 = -i\alpha^2 b^2 \int_0^y w_1 (2y)\, dy$$

Since $w_1 = w_1(y)$, we have upon integration by parts

$$\int_0^y w_1 (2y)\, dy = w_1(y^2) - \int_0^y y^2 \frac{dw_1}{dy}\, dy$$

Thus, equation 10-6 may be written in the form

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2(1-b^2) w_1 + i^3\alpha^2 b^2 \int_0^y y^2 \frac{dw_1}{dy}\, dy$$

$$= -\frac{A_1 R^2}{\mu} \qquad (10\text{-}7)$$

The complete solution of equation 10-7 is the sum of a particular integral and the complementary function. A particular integral of equation 10-7 is (refer to equations 3-16 and 3-17 of section III)

$$w_1 = \left(-\frac{A_1 R^2}{\mu}\right)\left[\frac{1}{i^3\alpha^2(1-b^2)}\right] = \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)}$$

We now find the complementary function, i.e., the solution of equation 10-7

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2(1-b^2)w_1 + i^3\alpha^2 b^2 \int_0^y y^2 \frac{dw_1}{dy}\,dy = 0 \tag{10-8}$$

We write equation 10-8 in more convenient form by a change in the independent variable, y, according to

$$x = b\alpha i^{1/2}\, y^2 \tag{10-9}$$

and define a constant, γ, by

$$\gamma = b\alpha i^{1/2}\left(\frac{1}{b^2} - 1\right) .$$

We consider the fourth term on the left-hand side of equation 10-8 with

$$x = b\alpha i^{1/2}\, y^2,\ dx = b\alpha i^{1/2}(2y)\,dy,\ \frac{1}{dy} = \frac{2b\alpha i^{1/2} y}{dx} ,$$

and lower limit: when y = 0, x = 0. We formally write x = x when y = y for the upper limit of integration. The actual relationship is $x = b\alpha i^{1/2} y^2$ when y = y. Thus,

$$i^3\alpha^2 b^2 \int_0^y y^2 \frac{dw_1}{dy}\, dy$$

$$= i^3\alpha^2 b^2 \int_0^x \left(\frac{x}{i^{1/2}\alpha b}\right) \frac{dw_1}{dx} \left(2 i^{1/2}\alpha b y\right)\left(\frac{1}{2 i^{1/2}\alpha b y}\right) dx$$

$$= \frac{i^3\alpha^2 b^2}{i^{1/2}\alpha b} \int_0^x x\, \frac{dw_1}{dx}\, dx$$

and equation 10-8 may be written as

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2(1-b^2)w_1 + \frac{i^3\alpha^2 b^2}{i^{1/2}\alpha b}\int_0^x x\,\frac{dw_1}{dx}\,dx = 0$$

or

$$-\left(\frac{i^{1/2}\alpha b}{i^3\alpha^2 b^2}\right)\left[\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2(1-b^2)w_1\right] - \int_0^x x\,\frac{dw_1}{dx}\,dx = 0$$

(10-10)

In equation 10-10, the product

$$-\frac{b\alpha i^{1/2}}{i^3\alpha^2 b^2}\, i^3\alpha^2(1-b^2)w_1 = -b\alpha i^{1/2}\left(\frac{1-b^2}{b^2}\right)w_1 = -b\alpha i^{1/2}\left(\frac{1}{b^2}-1\right)w_1 = -\gamma w_1$$

Now we shall consider the derivative terms in equation 10-8. From the relation $\frac{dw_1}{dx} = \frac{dw_1}{dy} \cdot \frac{dy}{dx}$ we have $\frac{dw_1}{dy} = \frac{dw_1}{dx} \cdot \frac{dx}{dy}$ and

$$\frac{d^2w_1}{dy^2} = (\frac{dw_1}{dx}) \frac{d^2x}{dy^2} + (\frac{dx}{dy}) \frac{d^2w_1}{dx^2} \frac{dx}{dy} = \frac{dw_1}{dx} \frac{d^2x}{dy^2} + \frac{d^2w_1}{dx^2} (\frac{dx}{dy})^2$$

From $x = b\alpha i^{1/2} y^2$, $\frac{dx}{dy} = b\alpha i^{1/2} (2y)$ $\frac{d^2x}{dy^2} = b\alpha i^{1/2} (2)$ the quantity,

$\frac{d^2w_1}{dy^2} + \frac{1}{y} (\frac{dw_1}{dy})$, in equation 10-10 may therefore be written as

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\left(\frac{dw_1}{dy}\right) = \frac{dw_1}{dx}\left(\frac{d^2x}{dy^2}\right) + \frac{d^2w_1}{dx^2}\left(\frac{dx}{dy}\right)^2 + \frac{1}{y}\left(\frac{dw_1}{dy}\right)$$

$$= \frac{dw_1}{dx}\left(2i^{1/2}\alpha b\right) + \frac{d^2w_1}{dx^2}\left(2i^{1/2}\alpha b y\right)^2 + \frac{1}{y}\left(\frac{dw_1}{dx}\right)\left(\frac{dx}{dy}\right)$$

$$= \frac{dw_1}{dx}\left(2i^{1/2}\alpha b\right) + \frac{d^2w_1}{dx^2}\left(2i^{1/2}\alpha b y\right)^2 + \frac{1}{y}\left(\frac{dw_1}{dx}\right)\left(2i^{1/2}\alpha b y\right)$$

$$= 4i^{1/2}\alpha b \frac{dw_1}{dx} + 4i\alpha^2 b^2 y^2 \frac{d^2w_1}{dx^2}$$

$$= 4i^{1/2}\alpha b \frac{dw_1}{dx} + 4i\alpha^2 b^2\left(\frac{x}{i^{1/2}\alpha b}\right)\frac{d^2w_1}{dx^2}$$

$$= 4i^{1/2}\alpha b \frac{dw_1}{dx} + 4i^{1/2}\alpha b\, x \frac{d^2w_1}{dx^2}$$

and $$-\frac{i^{1/2}\alpha b}{i^{3}\alpha^{2}b^{2}}\left[\frac{d^{2}w_{1}}{dy^{2}}+\frac{1}{y}\frac{dw_{1}}{dy}\right]$$

$$=-\frac{i^{1/2}\alpha b}{i^{3}\alpha^{2}b^{2}}\left[4i^{1/2}\alpha b\,\frac{dw_{1}}{dx}+4i^{1/2}\alpha b\,x\,\frac{d^{2}w_{1}}{dx^{2}}\right]$$

$$=-\frac{4i\alpha^{2}b^{2}}{i^{3}\alpha^{2}b^{2}}\left[\frac{dw_{1}}{dx}+x\,\frac{d^{2}w_{1}}{dx^{2}}\right]$$

$$=4\left[\frac{dw_{1}}{dx}+x\,\frac{d^{2}w_{1}}{dx^{2}}\right]$$

$$=4\left[\frac{d}{dx}\left(x\,\frac{dw_{1}}{dx}\right)\right]$$

Thus equation 10-10 may be written as

$$4 \frac{d}{dx} [x \frac{dw_1}{dx}] - \gamma w_1 - \int_0^x x(\frac{dw_1}{dx})\, dx = 0 \tag{10-11}$$

Again, for convenience, we substitute

$$v = x \frac{dw_1}{dx}$$

in equation 10-11 and obtain

$$4 \frac{d}{dx} (v) - \gamma w_1 - \int_0^x v\, dx = 0 \tag{10-12}$$

To eliminate the integral sign in equation 10-12, we differentiate throughout with respect to x and obtain

$$4 \frac{d}{dx}\left(\frac{dv}{dx}\right) - \gamma\left(\frac{dw_1}{dx}\right) - v = 0$$

or

$$4 \frac{d^2 v}{dx^2} - \gamma\left(\frac{v}{x}\right) - v = 0$$

or

$$\frac{d^2 v}{dx^2} + \left(-\frac{1}{4} - \frac{\gamma}{4x}\right) v = 0 \tag{10-13}$$

This is Whittaker's form of the equation for the Confluent Hypergeometric Function. We compare equation 10-13 with the general form of the equation in Whittaker and Watson, page 337, 338.

$$\frac{d^2W}{dz^2} + \left\{ -\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right\} W = 0 \tag{10-14}$$

Identifying the symbols used in equations 10-13 and 10-14, we find that W = v, z = x, k = - (γ/4), m = 1/2 and the solution of equation 10-14 is of the form

$$M_{k,m}(z) = z^{\frac{1}{2}+m} \cdot e^{-z/2} \left\{ 1 + \frac{\frac{1}{2}+m-k}{1!\,(2m+1)} \cdot z \right.$$

$$\left. + \frac{\left(\frac{1}{2}+m-k\right)\left(\frac{3}{2}+m-k\right)}{2!\,(2m+1)(2m+2)} \cdot z^2 + \ldots \right\}$$

or

$$M_{k,m}(z) = z^{\frac{1}{2}+m} \cdot e^{-z/2} \cdot {}_1F_1\left(m+\frac{1}{2} \,;\, 2m+1 \,;\, z\right)$$

in the notation of Kummer modified by Barnes. Thus,

$$M_{-\frac{\gamma}{4},\frac{1}{2}}(x) = x e^{-x/2} \cdot {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right),$$

and therefore

$$\frac{dw_1}{dx} = C_1 e^{-x/2} \cdot {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right)$$

The second solution of equation 10-13 is not required, since $dw_1/dy = 0$ at $y = 0$. Therefore, the solution of equation 10-12 is

$$w_1 = C_2 + C_1 \int_0^x e^{-x/2} \cdot {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx$$

and the value of C_2 must be determined by substitution in equation 10-12. If this is done, we find that $C_2 = (4/\gamma)C_1$ and

$$w_1 = \left(\frac{4}{\gamma}\right) C_1 + C_1 \int_0^x e^{-x/2} \cdot {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx \tag{10-15}$$

Defining a new variable

$$Z_0 = Z_0(\gamma, x) = \frac{4}{\gamma} + \int_0^x e^{-x/2} \cdot {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx \tag{10-16}$$

by analogy with equation 2-18, the solution of equation 10-10 satisfying the boundary condition $w_1 = 0$ at $y = 1$ may be written as

$$w_1 = w_1(x) = w_1\left(i^{1/2}\alpha \ell y^2\right) = \frac{A_1 R^2}{i\mu\alpha^2(1-\ell^2)} \left\{1 - \frac{Z_0(\gamma, x)}{Z_0(\gamma, i^{1/2}\alpha\ell)}\right\} \tag{10-17}$$

The expression for the average velocity, $\bar{w}_1$, is

$$\bar{w}_1(y) = \bar{w}_1 = \int_0^y w_1(2y)\, dy$$

at any point $y = r/R$ along the radius Changing variables from y to x according to $x = b\alpha i^{1/2} y$, $dx = b\alpha i^{1/2}(2y)\, dy$, $dx/(2b\alpha i^{1/2} y) = dy$ and limits: $y = 0$, $x = 0$; $y = 1$, $x = b\alpha i^{1/2}(1)^2 = b\alpha i^{1/2}$, the average velocity, $\bar{w}_1$, taken over the entire cross section from $y = 0$ to $y = 1$ or from $x = 0$ to $x = b\alpha i^{1/2}$ is

$$\bar{w}_1 = \int_{x=0}^{x=i^{1/2}\alpha b} w_1(2y) \frac{dx}{2 i^{1/2} \alpha b y}$$

$$= \frac{1}{i^{1/2}\alpha b} \int_0^{i^{1/2}\alpha b} w_1\, dx$$

Combining equations

$$w_1 = \frac{A_1 R^2}{i \mu \alpha^2 (1 - b^2)} \left\{ 1 - \frac{Z_0(\gamma, x)}{Z_0(\gamma, i^{1/2}\alpha b)} \right] \qquad (10\text{-}17)$$

and $$\bar{w}_1 = \frac{1}{i^{1/2}\alpha b} \int_0^{i^{1/2}\alpha b} w_1\, dx$$

we have

$$\overline{\omega}_1 = \frac{1}{i^{1/2}\alpha b}\int_0^{i^{1/2}\alpha b} \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)}\left\{1 - \frac{Z_0(\gamma, x)}{Z_0(\gamma, i^{1/2}\alpha b)}\right\} dx$$

$$= \frac{1}{i^{1/2}\alpha b}\left[\frac{A_1 R^2}{i\mu\alpha^2(1-b^2)}\right]\int_0^{i^{1/2}\alpha b} dx$$

$$- \frac{1}{i^{1/2}\alpha b}\left[\frac{A_1 R^2}{i\mu\alpha^2(1-b^2)}\right]\int_0^{i^{1/2}\alpha b} \frac{Z_0(\gamma, x)}{Z_0(\gamma, i^{1/2}\alpha b)} dx$$

$$= \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)}\left\{1 - \frac{1}{i^{1/2}\alpha b}\int_0^{i^{1/2}\alpha b} \frac{Z_0(\gamma, x)}{Z_0(\gamma, i^{1/2}\alpha b)} dx\right\} \tag{10-18}$$

In the integral $\int_0^{b\alpha i^{1/2}} \frac{Z_0(\gamma, x)}{Z_0(\gamma, b\alpha i^{1/2})} dx$ appearing in equation 10-18,

we let

$$u = Z_0(\gamma, x) = \frac{4}{\gamma} + \int_0^x e^{-x/2} {}_1F_1\left(1 + \frac{\gamma}{4}, 2, x\right) dx$$

$$dv = \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)}\, dx$$

$$\frac{du}{dx} = \frac{d}{dx}\left[Z_0(\gamma, x)\right]$$

$$= e^{-x/2}\, {}_1F_1\left(1 + \frac{\gamma}{4}, 2, x\right)$$

From $$dv = \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)}\, dx$$

$$v = \frac{x}{Z_0(\gamma, i^{1/2}\alpha b)}$$

Thus, according to the formula for integration by parts, we obtain

$$Z_0(\gamma, x) \cdot \frac{x}{Z_0(\gamma, i^{1/2}\alpha b)}\Bigg]_0^{i^{1/2}\alpha b}$$

$$- \int_0^{i^{1/2}\alpha b} \frac{x}{Z_0(\gamma, i^{1/2}\alpha b)} \cdot e^{-x/2} \, {}_1F_1\left(1 + \frac{\gamma}{4}, 2, x\right) dx$$

$$= Z_0(\gamma, i^{1/2}\alpha b) \frac{i^{1/2}\alpha b}{Z_0(\gamma, i^{1/2}\alpha b)}$$

$$- \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)} \int_0^{i^{1/2}\alpha b} x \, e^{-x/2} \, {}_1F_1\left(1 + \frac{\gamma}{4}, 2, x\right) dx$$

$$= i^{1/2}\alpha b - \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)} \int_0^{i^{1/2}\alpha b} x \, e^{-x/2} \, {}_1F_1\left(1 + \frac{\gamma}{4}, 2, x\right) dx$$

$$\bar{w}_1 = \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)} \left\{ 1 - \frac{1}{i^{1/2}\alpha b} \left[i^{1/2}\alpha b - \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)} \int_0^{i^{1/2}\alpha b} x\, e^{-x/2} \, {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx \right] \right\}$$

$$= \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)} \left\{ \frac{1}{Z_0(\gamma, i^{1/2}\alpha b)} \cdot \frac{1}{i^{1/2}\alpha b} \int_0^{i^{1/2}\alpha b} x\, e^{-x/2} \, {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx \right\}$$

$$= \frac{A_1 R^2}{i\mu\alpha^2(1-b^2)} \cdot \frac{Z_1(\gamma, i^{1/2}\alpha b)}{Z_0(\gamma, i^{1/2}\alpha b)} \tag{10-19}$$

where
$$Z_1(\gamma, x) = \frac{1}{i^{1/2}\alpha b} \int_0^x x\, e^{-x/2} \, {}_1F_1\left(1+\frac{\gamma}{4}, 2, x\right) dx$$

THE INTERACTION VELOCITY WHEN THE STEADY STREAM VELOCITY IS EQUAL TO THE WAVE VELOCITY

As an approximation, we set $b = 1$ in the earlier relation $b^2 = 2\bar{w}_0/c$ when the damping is very small, i.e., when the axial velocity of the steady stream, w_0 ($\bar{w}_0 = \frac{1}{2} w_0$) is equal to the pulse wave velocity, c. Note that when $b = 1$, we cannot use the result obtained in equation 10-19 because of the factor $(1 - b^2)$ in the denominator. So we start with the original equation (10-10), set $b = 1$ and obtain

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2 \int_0^y y^2 \frac{dw_1}{dy}\, dy = -\frac{A_1 R^2}{\mu} \qquad (10\text{-}20)$$

We note that a particular integral of equation 10-20 is no more a constant, as was obtained earlier for equation 10-10 where a particular integral was $w_1 = \dfrac{A_1 R^2}{i\mu\alpha^2(1 - b^2)}$ = constant. If, instead of the earlier substitution, $x = b\alpha i^{1/2} y^2$, in equation 10-10 we now change the independent variable in equation 10-20 according to $x = \dfrac{\alpha i^{3/2}}{2} y^2$, we have the following results from equation 10-20. From the first two terms on the left-hand side of equation 10-20, note that with the change of variables, $x = \dfrac{\alpha i^{3/2}}{2} y^2$, $dx = (\dfrac{\alpha i^{3/2}}{2})(2y)\, dy$, we have

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} = 2 i^{3/2} \alpha \left[x \frac{d^2 w_1}{dx^2} + \frac{dw_1}{dx} \right]$$

$$= 2 i^{3/2} \alpha \left[\frac{d}{dx}\left(x \frac{dw_1}{dx} \right) \right]$$

For the third term on the left-hand side of 10-20 we have, with the change in variables,

$$i^3\alpha^2 \int_0^y y^2 \frac{d\omega_1}{dy}\, dy$$

$$= i^3\alpha^2 \int_0^x \left(\frac{2x}{i^{3/2}\alpha}\right) \frac{d\omega_1}{dx} \left(i^{3/2}\alpha y\right) \frac{dx}{\left(i^{3/2}\alpha y\right)}$$

$$= 2 i^{3/2} \alpha \int_0^x x \frac{d\omega_1}{dx}\, dx$$

For the limits of integration, we formally write x = x when y = y. The actual relationship is $x = \frac{\alpha i^{3/2}}{2} y^2$. Thus, equation 10-20 may be written as

$$\left(2 i^{3/2} \alpha\right) \frac{d}{dx}\left(x \frac{d\omega_1}{dx}\right) + 2 i^{3/2} \alpha \int_0^x x \frac{d\omega_1}{dx}\, dx = -\frac{A_1 R^2}{\mu}$$

or

$$\frac{d}{dx}\left(x \frac{d\omega_1}{dx}\right) + \int_0^x x \left(\frac{d\omega_1}{dx}\right) dx = -\frac{A_1 R^2}{2 i^{3/2} \alpha y} \qquad (10\text{-}21)$$

To simplify the form of equation 10-21, we write $x \frac{dw_1}{dx} = \frac{dV}{dx}$ and obtain

$$\frac{d}{dx}\left(\frac{dV}{dx}\right) + \int_0^x \frac{dV}{dx}\,dx = -\frac{A_1 R^2}{2 i^{1/2} \alpha \mu}$$

or
$$\frac{d^2V}{dx^2} + V = -\frac{A_1 R^2}{2 i^{3/2} \alpha \mu} \tag{10-22}$$

together with the initial conditions:

$$\left.\frac{dV}{dx}\right]_{x=0} = \left. x\frac{d\omega_1}{dx}\right]_{x=0} = 0$$

$$\left. V \right]_{x=0} = 0$$

The solution of equation 10-22 may be written in the form

$$V(x) = M \cos x + N \sin x - \frac{A_1 R^2}{2 i^{3/2} \alpha \mu}$$

Applying the above initial conditions, we find that

$$M = \frac{A_1 R^2}{2 i^{3/2} \alpha \mu}, \qquad N = 0$$

thus $$V(x) = -\frac{A_1 R^2}{2 i^{3/2} \alpha \mu} \left(1 - \cos x \right) \tag{10-23}$$

To determine w_1, we refer to the earlier substitution, $x \frac{dw_1}{dx} = \frac{dV}{dx}$, and obtain $w_1 = \int \frac{1}{x} \frac{dV}{dx} dx$. Substituting the expression for V(x) from above,

$$w_1 = -\frac{A_1 R^2}{2 i^{3/2} \alpha \mu} \left\{ \int_0^{\frac{1}{2} i^{3/2} \alpha} \left[\frac{1 - \cos x}{x} \right] dx - \int_0^{x} \left[\frac{1 - \cos x}{x} \right] dx \right\} \tag{10-24}$$

To obtain the average velocity, $\bar{w}_1$, we start with the earlier relationship $\bar{w}_1 = \frac{1}{b\alpha i^{1/2}} \int_0^{\alpha i^{1/2}} w_1 \, dx$, substitute the value of w_1 obtained above and find that

$$\bar{w}_1 = \frac{A_1 R^2}{i \mu \alpha^2} \left\{ \frac{i^{3/2} \alpha}{2} - \sin \frac{i^{3/2} \alpha}{2} \right\} \tag{10-25}$$

This remarkably simple result, obtained by assuming b = 1, i.e., $2\bar{w}_0 = c$, is not likely to have any practical application to arterial flow. The only place where the steady stream velocity could approach half the pulse velocity, $2\bar{w}_0 \to c$, would be in the thoracic aorta, where inlet conditions, and possible turbulence, might well nullify the entire theory. Moreover, since c is always complex, the condition b = 1 can never exist, except as an approximation when the damping is very small.

APPROXIMATE SOLUTION IN TERMS OF BESSEL FUNCTIONS

This approximation considers the steady stream velocity small as compared with the pulse-wave velocity. This approximation does not compare the relative magnitudes of the velocities of the steady stream and the oscillatory flow. Accordingly, it is desirable to check the limiting form of the solution of equation 10-18 for $b = (2\bar{w}_0/c) \to 0$; i.e., for

$$\gamma = b\alpha i^{1/2}\left(\frac{1}{b^2} - 1\right) = \alpha i^{1/2}\left(\frac{1}{b} - b\right) \to \infty.$$

When γ is large, equation 10-13 reduces to the normal form of Bessel's equation

$$\frac{d^2v}{dx^2} - \frac{\gamma}{4}\left(\frac{1}{x}\right) v = 0 \qquad (10\text{-}26)$$

with the solution $v = x^{1/2} J_1(i\gamma^{1/2} x^{1/2})$. Rewriting this result in terms of y as an independent variable, we obtain

$$\frac{dw_1}{dy} = C_1 J_1\left(\sqrt{1-b^2}\,\alpha\, i^{3/2}\, y\right) \qquad (10\text{-}27)$$

which reduces to the solution already known for b = 0. The simplicity of this equation suggests that we examine it for the range of values of b, over which it would be a reasonable approximation, i.e., for what values of γ is the inequality, $\gamma/4x \gg 1$, valid. From the relation $x = b\alpha i^{1/2} y^2$, we note that the maximum value of $y = r/R$ is 1. It follows that the maximum value of x is $b\alpha i^{1/2}$. Thus the inequality $\gamma/4x \gg 1$ with $x = b\alpha i^{1/2}$ and

$\gamma = b\alpha i^{1/2}\left(\frac{1}{b^2} - 1\right)$ becomes

$$\frac{1}{4}\left(\frac{1}{b^2} - 1\right) \gg 1$$

Substituting $b^2 = \frac{2\bar{w}_0}{c}$, we have

$$\frac{1}{4}\left(\frac{c}{2\,\bar{w}_0} - 1\right) \gg 1$$

or

$$\frac{\bar{w}_0}{c} \ll \frac{1}{10}$$

In the femoral artery, McDonald's results give $\bar{w}_0 = 15$ cm/sec. The pulse velocity, c, is not less than 450 cm/sec. Thus, for these conditions, $\bar{w}_0/c \approx 1/30$. However, nearer the heart it is to be expected that $\bar{w}_0$ would be greater than 15 cm/sec and the pulse velocity, c, less than 450 cm/sec. Therefore, practical conditions seem, at best, to be beyond the range of usefulness of the approximation. This is unduly pessimistic, for a closer approximation may be obtained very simply by applying the method used in section IX.

Consider the earlier equation

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\alpha^2(1-b^2)w_1 + i^3\alpha^2 b^2 \int_0^y y^2 \frac{dw_1}{dy}\,dy = -\frac{A_1 R^2}{\mu} \tag{10-10}$$

We set $\beta^2 = \alpha^2(1 - b^2)$ and write (10-10) in the form

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\beta^2 w_1 = -\frac{A_1 R^2}{\mu} - i^3\alpha\beta^2 \int_0^y y^2 \frac{dw_1}{dy}\,dy \tag{10-28}$$

Equation 10-28 includes the effect of the oscillatory component exhibited by the second term on the right-hand side. The first term on the right-hand side is the steady stream component. Note that both the terms on the right-hand side of equation 10-28 have the same algebraic sign. Thus, for a small steady stream flowing in the direction of travel of the pulse wave, the amount of oscillatory flow is increased. The total flow is reduced if the steady stream flows in the opposite direction. If we drop the second term on the right-hand side in equation 10-28, we obtain

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} + i^3\beta^2 w_1 = -\frac{A_1 R^2}{\mu}$$

This equation is the same as equation 2-4, with β replacing α. Referring to equations 2-4 and 2-5, we note that we may write down its solution as

$$w_1 = \frac{A_1 R^2}{i\mu\beta^2}\left\{1 - \frac{J_0(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\right\} \tag{10-29}$$

We consider equation 10-29 as the first approximate solution of equation 10-28.

In equation 10-28, since $b^2 = (2\bar{w}_0/c) < 1$, i.e., the pulse wave velocity, c, is much larger than the stream velocity, w_0, the accuracy of the first approximate solution, equation 10-29, can be improved by taking the first approximate form for w_1, namely,

$$w_1 = \frac{A_1 R^2}{i\mu\beta^2}\left\{1 - \frac{J_0\left(i^{3/2}\beta y\right)}{J_0\left(i^{3/2}\beta\right)}\right\}$$

forming

$$\frac{dw_1}{dy} = \frac{A_1 R^2}{i\mu\beta^2}\left(\beta i^{3/2}\right)\frac{J_1\left(i^{3/2}\beta y\right)}{J_0\left(i^{3/2}\beta\right)}$$

and substituting on the right-hand side of equation 10-28. Accordingly, we note that

$$-i^3 b^2 \alpha^2 \int_0^y y^2 \frac{dw_1}{dy}\, dy$$

$$= -i^3 b^2 \alpha^2 \left(\frac{A_1 R^2}{i\mu\beta^2}\right)\beta i^{3/2}\int_0^y y^2\, \frac{J_1\left(i^{3/2}\beta y\right)}{J_0\left(i^{3/2} y\right)}\, dy$$

$$= -i^3 b^2 \alpha^2 \left(\frac{A_1 R^2}{i \mu \beta^2} \right) \beta i^{3/2} \left\{ \left(\frac{y^2}{i^{3/2}\beta} \right) \frac{J_2 (i^{3/2} \beta y)}{J_0 (i^{3/2}\beta)} \right\}$$

$$= \frac{b^2 \alpha^2}{\beta^2} \left(\frac{A_1 R^2}{\mu} \right) \left\{ y^2 \frac{J_2 (i^{3/2} \beta y)}{J_0 (i^{3/2}\beta)} \right\}$$

$$= \left(\frac{b^2}{1-b^2} \right) \left(\frac{A_1 R^2}{\mu} \right) \left\{ y^2 \frac{J_2 (i^{3/2} \beta y)}{J_0 (i^{3/2}\beta)} \right\}$$

If we write w_2 for the correction due to the presence of the oscillatory components in w_1, we may write the equation for w_2 in the form

$$\frac{d^2 w_2}{dy^2} + \frac{1}{y} \frac{dw_2}{dy} + i^3 \beta^2 w_2$$

$$= \left(\frac{b^2}{1-b^2} \right) \frac{A_1 R^2}{\mu} \left\{ y^2 \frac{J_2 (i^{3/2} \beta y)}{J_0 (i^{3/2}\beta)} \right\} \qquad (10\text{-}31)$$

Note that we do not write the term $(-A_1R^2)/\mu$ on the right side of the above equation, since we have already taken care of this in the first approximate solution, equation 10-29.

It is not necessary to solve equation 10-31 for w_2 and then determine $\bar{w}_2$ because we can directly obtain an expression describing $\bar{w}_2$ as follows. We note that the solution of the earlier equation

$$\frac{d^2 q}{dy^2} + \frac{1}{y}\frac{dq}{dy} + i^3\alpha^2 q = g(y)$$

where
$$g(y) = \int_0^y f(y)\, 2y\, dy$$

and f(y) is a known function of y, is

$$q(y)\Big]_{y=1} = \frac{1}{i^3\alpha^2}\int_0^1 \left\{1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)}\right\} f(y)\, 2y\, dy$$

By analogy, the solution of equation 10-31 may be written as

$$\bar{w}_2 = \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1R^2}{\mu}\right)\left(\frac{1}{i^3\beta^2}\right)\int_0^1 \left\{1 - \frac{J_0(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\right\}\left\{y^2\frac{J_2(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\right\}(2y)\, dy$$

$$= \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1R^2}{\mu}\right)\left(\frac{1}{i^3\beta^2}\right)\left[\int_0^1 (2y^3)\frac{J_2(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\, dy - \int_0^1 (2y^3)\frac{J_2(i^{3/2}\beta y)\, J_0(i^{3/2}\beta y)}{[J_0(i^{3/2}\beta)]^2}\, dy\right] \tag{10-32}$$

$$= \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1R^2}{\mu}\right)\left(\frac{1}{i^3\beta^2}\right)\left[\left(\frac{1}{i^{3/2}\beta}\right)(2y^3)\,\frac{J_3(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\,\Big|_0^1\right.$$

$$\left. - \int_0^1 (2y^3)\,\frac{J_2(i^{3/2}\beta y)\,J_0(i^{3/2}\beta y)}{\left[J_0(i^{3/2}\beta)\right]^2}\,dy\right]$$

$$= \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1R^2}{i^3\beta^2\mu}\right)\left(\frac{2}{i^{3/2}\beta}\right)\frac{J_3(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}$$

$$- \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1R^2}{i^3\beta^2\mu}\right)\int_0^1 (2y^3)\,\frac{J_2(i^{3/2}\beta y)\,J_0(i^{3/2}\beta y)}{\left[J_0(i^{3/2}\beta)\right]^2}\,dy \qquad (10\text{-}33)$$

In equation 10-33 we use the well-known result

$$\int z^3 J_2(z)\, J_0(z)\, dz = \frac{z^4}{6}\left\{ J_2(z)\, J_0(z) + J_3(z)\, J_1(z) \right\}$$

and obtain

$$\bar{\omega}_2 = \left(\frac{b^2}{1-b^2}\right)\left(\frac{A_1 R^2}{i^3 \beta^2 \mu}\right)\left[\left(\frac{2}{i^{3/2}\beta}\right) \frac{J_3(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}\right.$$

$$- \frac{y^4}{3}\left\{ \frac{J_2(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)} \cdot \frac{J_0(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)}\right.$$

$$\left.\left. + \frac{J_1(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)} \cdot \frac{J_3(i^{3/2}\beta y)}{J_0(i^{3/2}\beta)} \right\} \right]$$

The value of $\bar{w}_2$ at the tube wall is obtained by setting $y = (r/R) = 1$ in the above equation. Thus

$$\bar{\omega}_2\Big]_{y=1} = \left(\frac{\ell^2}{1-\ell^2}\right)\left(\frac{A_1R^2}{i^3\beta^2\mu}\right)\left[\left(\frac{2}{i^{3/2}\beta}\right)\frac{J_3(i^{3/2}\beta)}{J_0(i^{3/2}\beta)} - \left(\frac{1}{3}\right)\frac{J_2(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}\right.$$

$$\left. - \left(\frac{1}{3}\right)\frac{J_1(i^{3/2}\beta)}{J_0(i^{3/2}\beta)} \cdot \frac{J_3(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}\right] \tag{10-34}$$

Expressing J_3 in terms of J_2 and J_1 by means of the recurrence formula

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z)$$

we have for $n = 2$

$$J_3(z) = \frac{4}{z} J_2(z) - J_1(z)$$

Accordingly, the bracketed portion of equation 10-34 may be written as

$$\left(\frac{2}{i^{3/2}\beta}\right)\left(\frac{1}{J_0(i^{3/2}\beta)}\right)\left[\left(\frac{4}{i^{3/2}\beta}\right)J_2(i^{3/2}\beta) - J_1(i^{3/2}\beta)\right]$$

$$-\left(\frac{1}{3}\right)\frac{J_2(i^{3/2}\beta)}{J_0(i^{3/2}\beta)} - \left(\frac{1}{3}\right)\frac{J_1(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}\left(\frac{1}{J_0(i^{3/2}\beta)}\right)\left[\left(\frac{4}{i^{3/2}\beta}\right)J_2(i^{3/2}\beta) - J_1(i^{3/2}\beta)\right]$$

$$= \left(\frac{2}{i^{3/2}\beta}\right) \frac{1}{J_0} \left[\left(\frac{4}{i^{3/2}\beta}\right) J_2 - J_1\right] - \left(\frac{1}{3}\right) \frac{J_2}{J_0}$$

$$- \left(\frac{1}{3}\right) \frac{J_1}{J_0^2} \left[\left(\frac{4}{i^{3/2}\beta}\right) J_2 - J_1\right]$$

$$= \left(\frac{8}{i^3\beta^2}\right) \frac{J_2}{J_0} - \left(\frac{2}{i^{3/2}\beta}\right) \frac{J_1}{J_0} - \left(\frac{1}{3}\right) \frac{J_2}{J_0} - \left(\frac{4}{3\, i^{3/2}\beta}\right) \frac{J_1 J_2}{J_0^2}$$

$$+ \left(\frac{1}{3}\right) \frac{J_1^2}{J_0^2}$$

$$= \left\{ \left(\frac{4}{i^{3/2}\beta} \frac{J_2}{J_0} - \frac{J_1}{J_0}\right)\left(\frac{2}{i^{3/2}\beta} - \frac{1}{3} \frac{J_1}{J_0}\right) - \frac{1}{3} \frac{J_2}{J_0} \right\} \qquad (10\text{-}35)$$

Thus we may write

$$\bar{\omega}_2\Big]_{y=1} = \left(\frac{\ell}{1-\ell^2}\right)\left(\frac{A_1 R^2}{i^3\beta^2\mu}\right)\left[\left(\frac{4}{i^{3/2}\beta}\frac{J_2}{J_0} - \frac{J_1}{J_0}\right)\left(\frac{2}{i^{3/2}\beta} - \frac{1}{3}\frac{J_1}{J_0}\right) - \frac{1}{3}\frac{J_2}{J_0}\right]$$

In the above equation the expression in brackets may be further reduced by expressing J_2 in terms of J_1 and J_0 according to the relation

$$J_2(z) = \frac{2}{z}J_1(z) - J_0(z)$$

and introducing the modulus and phase form for J_1/J_0. We note that

$$M_{10}'(\alpha) = \left| 1 - \frac{2 J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)} \right|$$

$$\varepsilon_{10}'(\alpha) = \text{phase}\left(1 - \frac{2 J_1(i^{3/2}\alpha)}{i^{3/2}\alpha\, J_0(i^{3/2}\alpha)} \right)$$

$$\left(-\frac{8}{i^3\beta^2}\right)\frac{J_1}{J_0} = \left(-\frac{8}{i^3\beta^2}\right) M_{10}'(\alpha)\, e^{i\,\varepsilon_{10}'(\alpha)}$$

$$-M_{10}'(\alpha)e^{i\varepsilon_{10}'(\alpha)} \cdot \left(\frac{2}{3}\right)M_{10}'(\alpha)e^{i\varepsilon_{10}'(\alpha)} = -\left(\frac{2}{3}\right)\left(\frac{J_1}{J_0}\right)^2$$

$$2M_{10}'(\alpha)e^{i\varepsilon_{10}'(\alpha)} = (2)\frac{J_1}{J_0}$$

Using the relation $J_2(z) = \frac{2}{z} J_1(z) - J_0(z)$, the form above

$$\left(\frac{8}{i^3\beta^2}\right)\frac{J_2}{J_0} - \left(\frac{2}{i^{3/2}\beta}\right)\frac{J_1}{J_0} - \left(\frac{1}{3}\right)\frac{J_2}{J_0} - \left(\frac{4}{3i^{3/2}\beta}\right)\frac{J_1J_2}{J_0^2} + \left(\frac{1}{3}\right)\frac{J_1^2}{J_0^2}$$

may be writte.. as

$$\left(\frac{8}{i^3\beta^2}\right)\frac{1}{J_0}\left(\frac{2}{i^{3/2}\beta}J_1 - J_0\right) - \left(\frac{2}{i^{3/2}\beta}\right)\frac{J_1}{J_0} - \left(\frac{1}{3}\right)\left(\frac{1}{J_0}\right)\left(\frac{2}{i^{3/2}\beta}J_1 - J_0\right)$$

$$-\left(\frac{4}{3i^{3/2}\beta}\right)\frac{J_1}{J_0^2}\left(\frac{2}{i^{3/2}\beta}J_1 - J_0\right) + \frac{1}{3}\left(\frac{J_1}{J_0}\right)^2$$

$$= \left(\frac{16}{i^{9/2}\beta^3}\right)\frac{J_1}{J_0} - \frac{8}{i^3\beta^2} - \left(\frac{2}{i^{3/2}\beta}\right)\frac{J_1}{J_0} - \left(\frac{2}{3i^{3/2}\beta}\right)\frac{J_1}{J_0}$$

$$+ \frac{1}{3} - \left(\frac{8}{3i^3\beta^2}\right)\left(\frac{J_1}{J_0}\right)^2 + \left(\frac{4}{3i^{3/2}\beta}\right)\frac{J_1}{J_0} + \left(\frac{1}{3}\right)\left(\frac{J_1}{J_0}\right)^2$$

$$= \left\{\left[\left(\frac{8}{i^3\beta^2} - 1\right)\frac{J_1}{J_0} - \frac{4}{i^{3/2}\beta}\right]\left(\frac{2}{i^{3/2}\beta} - \frac{1}{3}\frac{J_1}{J_0}\right) - \frac{2}{3i^{3/2}\beta}\frac{J_1}{J_0} + \frac{1}{3}\right\}$$

If, for convenience, we write

$$\frac{2J_1\left(i^{3/2}\beta\right)}{i^{3/2}\beta\, J_0\left(i^{3/2}\beta\right)} \equiv L \ ,$$

$$M'_{10}(\beta) = \left|1 - \frac{2J_1\left(i^{3/2}\beta\right)}{i^{3/2}\beta\, J_0\left(i^{3/2}\beta\right)}\right| = |1 - L|$$

$$1 - L = M'_{10}(\beta)\, e^{i\varepsilon'_{10}(\beta)}$$

then the last form above may be written as

$$\left[\frac{4L}{i^{3/2}\beta} - \frac{i^{3/2}\beta}{2}L\right]\left[\frac{2}{i^{3/2}\beta} - \frac{i^{3/2}\beta}{(2)(3)}L\right] - \frac{8}{i^{3}\beta^{2}} + \frac{4}{(2)(3)}L - \frac{L}{3} + \frac{1}{3}$$

It can be shown that the expression

$$\left[\frac{4}{i^{3/2}\beta}\left(\frac{J_2}{J_0}\right) - \frac{J_1}{J_0}\right]\left[\frac{2}{i^{3/2}\beta} - \left(\frac{1}{3}\right)\frac{J_1}{J_0}\right] - \left(\frac{1}{3}\right)\frac{J_2}{J_0} \tag{10-35}$$

is equivalent to the expression

$$-M'_{10}e^{i\varepsilon'_{10}}\left\{\frac{8}{i^{3}\beta^{2}} + \left(\frac{2}{3}\right)M'_{10}e^{i\varepsilon'_{10}} - 2\right\} - 1$$

$$+ \frac{1}{3}\left\{\frac{J_1(i^{3/2}\beta)}{J_0(i^{3/2}\beta)}\right\}^2 \tag{10-36}$$

If there were no damping of the pulse wave in transmission, then the pulse-wave velocity, c, would be real and the effect of substituting β for α in the Bessel functions could be calculated from the available tables. Since c is complex, the Bessel functions are no longer functions of $i^{3/2}$, but of a general complex argument.

If c_1 is the measured velocity of the pulse wave, then

$$b^2 = \frac{2\bar{w}_0}{c} = \frac{2\bar{w}_0}{c_1} \cdot \frac{c_1}{c} = \frac{2\bar{w}_0}{c_1} \cdot \frac{c_1}{c} \cdot \frac{c_0}{c_0} = \frac{2\bar{w}_0}{c_1} \cdot \left(\frac{c_0}{c}\right) \cdot \left(\frac{c_1}{c_0}\right)$$

$$= \frac{2\bar{w}_0}{c_1} \cdot (X - iY)\left(\frac{1}{X}\right) = \frac{2\bar{w}_0}{c_1}\left\{1 - i\,\frac{Y}{X}\right\}$$

Now $\beta^2 = \alpha^2(1 - b^2)$ and if we write

$$\beta_0{}^2 = \alpha^2\left(1 - \frac{2\bar{w}_0}{c_1}\right)$$

then $$\frac{\beta^2}{\beta_0^2} = \frac{\alpha^2(1-b^2)}{\alpha^2\left(1-\frac{2\bar{w}_0}{c_1}\right)} = (1-b^2)\left[\frac{1}{1-\frac{2\bar{w}_0}{c_1}}\right]$$

$$= (1-b^2)\left[1+\frac{2\bar{w}_0}{c_1}\right]$$ to first order in $2\bar{w}_0/c_1$

$$= 1+\frac{2\bar{w}_0}{c_1} - b^2 - b^2\left(\frac{2\bar{w}_0}{c_1}\right)$$

$$= 1+\frac{2\bar{w}_0}{c_1} - \frac{2\bar{w}_0}{c_1}\left(1-i\frac{Y}{X}\right) - \frac{2\bar{w}_0}{c_1}\left(1-i\frac{Y}{X}\right)\left(\frac{2\bar{w}_0}{c_1}\right)$$

$$= 1+\frac{2\bar{w}_0}{c_1} - \frac{2\bar{w}_0}{c_1} + i\left(\frac{2\bar{w}_0}{c_1}\right)\frac{Y}{X}$$ + terms containing c_1^2 in

in the denominator to be neglected.

Thus, $$\frac{\beta^2}{\beta_0^2} = 1 + i\left(\frac{2\bar{\omega}_0}{c_1}\right)\frac{Y}{X}$$ to first order in $2\bar{w}_0/c_1$,

and $$\beta = \beta_0\left[1 + i\left(\frac{2\bar{\omega}_0}{c_1}\right)\frac{Y}{X}\right]^{1/2}$$

If the quantity $2\bar{w}_0/c$ is small, it is possible to derive an approximation in terms of known functions by using the well-known formula

$$J_n(\lambda z) = \lambda^n \sum_{m=0}^{\infty} J_{n+m}(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^m \tag{10-37}$$

From equation 10-37 we have

for $n = 0$, $$J_0(\lambda z) = \sum_{m=0}^{\infty} J_m(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^m$$

for n = 1, $$J_1(\lambda z) = \lambda \sum_{m=0}^{\infty} J_{m+1}(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^m$$

$$= \lambda \left\{ J_1(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^0 + \sum_{m=1}^{\infty} J_m(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^m \right\}$$

$$= \lambda \left\{ J_1(z) + \sum_{m=1}^{\infty} J_m(z)\left[\left(\frac{1-\lambda^2}{2}\right)z\right]^m \right\}$$

In the earlier relation

$$\beta = \beta_0\left[1 + i\left(\frac{2\bar{\omega}_c}{c_1}\right)\frac{Y}{X}\right]^{1/2}$$

we write

$$\lambda^2 = 1 + i\left(\frac{2\bar{\omega}_0}{c_1}\right)\frac{Y}{X}$$

Thus, $\beta = \beta_0\lambda$, $\lambda = \beta/\beta_0$, and $\lambda z = (\frac{\beta}{\beta_0})z$. Moreover, $1 - \lambda^2 = i\,\frac{2\bar{w}_0}{c_1}\,\frac{Y}{X}$, and $(\frac{1 - \lambda^2}{2})z^2 = i(\frac{\bar{w}_0}{c_1})(\frac{Y}{X})z^2$.

Now we consider the term $M_{10}'(\beta)\, e^{i\varepsilon_{10}'(\beta)}$ appearing in the expression (10-36). We note that

$$M_{10}(\beta)\, e^{i\varepsilon_{10}'(\beta)} = 1 - \frac{2 J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)}$$

We wish to write the right-hand side of this equation in terms of the parameter β_0.

From the relation $\beta = \beta_0 \left[1 + i \left(\frac{2\bar{\omega}_c}{c_1} \right) \frac{Y}{X} \right]^{1/2}$

we note that when the imaginary part $Y = 0$, i.e., when damping of the pulse-wave is absent and c is real, $\beta = \beta_0$. Thus, when $Y = 0$,

$$\frac{J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)} = \frac{J_1(i^{3/2}\beta_0)}{i^{3/2}\beta_0\, J_0(i^{3/2}\beta_0)}$$

We may consider the term $\dfrac{J_1(i^{3/2}\beta_0)}{i^{3/2}\beta_0\, J_0(i^{3/2}\beta_0)}$

as the first term in the expansion of $\dfrac{J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)}$ in powers of Y where Y is small. The other terms in the expansion of $\dfrac{J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)}$ are given in equation 10-38.

For convenience, let us denote the function of $\beta i^{3/2}$,

$f(\beta i^{3/2}) = \frac{J_1(\beta i^{3/2})}{\beta i^{3/2} J_0(\beta i^{3/2})}$. It follows that

$$f'(i^{3/2}\beta) = \frac{M'_{10}(\beta)\, e^{i\varepsilon'_{10}(\beta)} + \left(\frac{J_1}{J_0}\right)^2}{i^{3/2}\beta}$$

Consider the Taylor series expansion $f(x) = f(a) + f'(a)[x - a]$

where $$f(x) = 1 - \frac{2 J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)}$$

$$f(a) = 1 - \frac{2 J_1(i^{3/2}\beta_0)}{i^{3/2}\beta_0\, J_0(i^{3/2}\beta_0)}$$

$$f'(a) = \left(\frac{1}{i^{3/2}\beta_0}\right)\left[M'_{10}(\beta_0)\, e^{i\varepsilon'_{10}(\beta_0)} + \left\{\frac{J_1(i^{3/2}\beta_0)}{J_0(i^{3/2}\beta_0)}\right\}^2\right]$$

$$(x - a) = i^{3/2}\beta - i^{3/2}\beta_0$$

Thus, the expansion of $1 - \frac{2J_1(\beta i^{3/2})}{\beta i^{3/2} J_0(\beta i^{3/2})}$ may be written in the form

$$1 - \frac{2J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)} = 1 - \frac{2J_1(i^{3/2}\beta_0)}{i^{3/2}\beta_0\, J_0(i^{3/2}\beta_0)}$$

$$+ \left(\frac{\beta}{\beta_0} - 1\right)\left[M_{10}'(\beta_0)\, e^{i\epsilon_{10}'(\beta_0)} + \left\{\frac{J_1(i^{3/2}\beta_0)}{J_0(i^{3/2}\beta_0)}\right\}^2 \right]$$

Now, $$\frac{\beta}{\beta_0} = \left[1 + i\left(\frac{2\bar{\omega}_0}{c_1}\right)\frac{Y}{X}\right]^{1/2}$$

$$= 1 + \frac{i\bar{\omega}_0}{c_1}\left(\frac{Y}{X}\right)$$ to first order,

and $$\frac{\beta}{\beta_0} - 1 = \frac{i\bar{\omega}_0}{c_1}\left(\frac{Y}{X}\right)$$

Putting all these facts together, we write

$$1 - \frac{2 J_1(i^{3/2}\beta)}{i^{3/2}\beta\, J_0(i^{3/2}\beta)} = 1 - \frac{2 J_1(i^{3/2}\beta_0)}{i^{3/2}\beta_0\, J_0(i^{3/2}\beta_0)}$$

$$+ \, i\,\frac{\bar{w}_0}{c_1}\left(\frac{Y}{X}\right)\left[M_{10}'(\beta_0)e^{i\varepsilon_{10}'(\beta_0)} + \left\{\frac{J_1(i^{3/2}\beta_0)}{J_0(i^{3/2}\beta_0)}\right\}^2\right] \qquad (10\text{-}38)$$

We shall now obtain an expression for the average fluid velocity, $\bar{w}$, and correct to first order in $2\bar{w}_0/c$ by combining the two expressions (10-36) and (10-38). Since the expression (10-36) is itself a first-order correction, it will be sufficiently accurate to write β_0 for β in it. Thus the corrected average fluid velocity may be written in the form

$$\bar{w} = \bar{w}_1 + \bar{w}_2 = \frac{A_1 R^2}{i\beta_0^2 \mu}\left\{M_{10}'(\beta_0)e^{i\varepsilon_{10}'(\beta_0)} + \left(\frac{2\bar{w}_0}{c_1}\right)S_{10}(\beta_0)\right\} \qquad (10\text{-}39)$$

where

$$S_{10}(\beta_0) = 1 + M_{10}'e^{i\varepsilon_{10}'}\left(\frac{8i}{\beta_0^2} + \frac{2}{3}M_{10}'e^{i\varepsilon_{10}'} - 2\right)$$

$$-\frac{1}{3}\left\{\frac{J_1(i^{3/2}\beta_0)}{J_0(i^{3/2}\beta_0)}\right\}^2 - i\tan\left(\frac{\varepsilon_{10}'}{2}\right)\left[M_{10}'e^{i\varepsilon_{10}'} + \left\{\frac{J_1(i^{3/2}\beta_0)}{J_0(\beta_0 i^{3/2})}\right\}^2\right]$$

(10-40)

Table XI is a table of $S_{10}(\beta_0)$ for $0 \leqq \beta_0 \leqq 10$ at intervals of 0.05. The first four columns give the real and imaginary parts, and the modulus and phase of $S_{10}(\beta_0)$, in that order. The last two columns are the real and imaginary parts of $1 - \frac{2J_1(\beta_0 i^{3/2})}{\beta_0 J_0(\beta_0 i^{3/2})}$ i.e., the quantities C_m and D_m of section II, equations 2-55 and 2-56.

For the experimental results of McDonald, which have been previously used as an example, the steady component of the average velocity is 15 cm/sec. Only part of this, however, is generated by the steady component of the pressure gradient. As was shown in section IX, 1.7 cm/sec of this steady component of the average velocity is caused by interactions between the harmonic terms, leaving 13.3 cm/sec generated by the pressure gradient. If we assume the pulse velocity to be 450 cm/sec, we obtain

$$\frac{2\bar{w}_0}{c_1} = 0.06 = b^2 \quad \text{and} \quad \frac{1}{1 - b^2} = \frac{1}{0.94} \; .$$

From a succession of trial values of β_0, therefore, we calculate the real and imaginary parts of the expression

$$\frac{1}{0.94} \{1 - F_{10}(\beta_0) + 0.06 \; S_{10}(\beta_0)\}$$

from table VIII of Womersley (1958) for each of the four harmonics. Using these real and imaginary parts as our values of C_m and D_m in equation 2-57, we find the value of β_0 for which the combined oscillatory terms will be equal and opposite to the steady velocity at the observed point of flow reversal. In McDonald's experiment, this observed point was at 125° of the cycle. The best fit at this observed point was given by $\beta_0 = 2.5$, corresponding to $\alpha = 2.58$. The coefficients of the Fourier series for $\bar{w}$ are given in table XI.

In figure 56 we compare these calculated values with McDonald's observed values. The ordinates in this figure are in flow units, obtained by multiplying the coefficients in table XI by the cross-sectional area, the value of R taken being the same as that assumed by McDonald, i.e., R = 1.5 cm. The fit to the observations is not improved much by using the "exact" solution. Except for a slight increase in diastolic flow, as good a fit can be obtained by using the simple theory with $\alpha = 2.7$. See figure 56.

TABLE XI

THE COEFFICIENTS OF THE FOURIER SERIES FOR $\bar{w}$ CORRESPONDING TO THE FIRST FOUR HARMONICS

Harmonic	Coefficient of cos nt	Coefficient of sin nt
1	21.81	20.74
2	-25.97	18.81
3	-9.73	-8.66
4	-0.28	-3.18

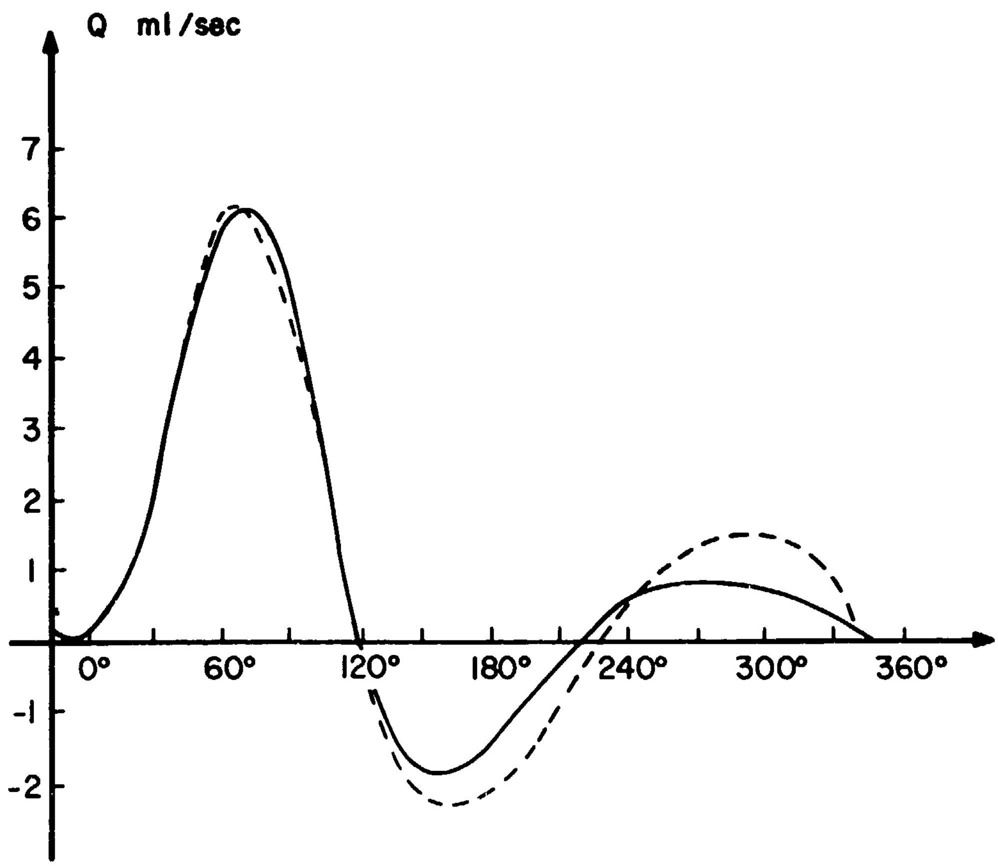

Figure 56. Variation in Flow Over One Cardiac Cycle in the Femoral Artery of the Dog.

Full Line: Calculated from the first-order approximation to the "exact" solution with $\alpha = 2.5$.

Broken Line: McDonald's observed values.

THE APPROXIMATE SOLUTION AND THE FREQUENCY EQUATION

If we substitute the approximate solution for the corrected average velocity, $\bar{w}$, in the frequency equation, the reduced determinant would have the form

$$\begin{vmatrix} 1 - \psi_{10}(\beta_0) & 2 & 1 \\ 1 & x & 1 + \sigma x \\ -i\Theta - \frac{1}{2}\psi_{10}(\beta_0) & -\sigma x & k - x + i\Theta \end{vmatrix} = 0 \tag{10-41}$$

where
$$1 - \psi_{10}(\beta_0) = 1 - F_{10}(\beta_0) + \left(\frac{2\bar{w}_0}{c_1}\right) S_{10}(\beta_0) \tag{10-42}$$

and
$$\Theta = \frac{1}{\beta^2}\left(\frac{A_0}{A_1}\right)\frac{4\bar{w}_0}{c} \tag{10-43}$$

The last term, Θ, above, represents the viscous drag due to the steady stream. Note that the value of Θ is small over the range of values of α which are of interest. For example, in the femoral artery,

$$\alpha^2 \simeq 7, \ \frac{A_0}{A_1} \simeq \frac{1}{8} \ \text{ and } \ \frac{\bar{w}_0}{c} \simeq 0.03$$

so that $\Theta < \frac{1}{400}$. Even in the thoracic aorta, assuming the values $\frac{A_0}{A_1} = 1$, $\frac{\bar{w}_0}{c} = \frac{1}{4}$, we find that $\Theta < \frac{1}{100}$, since $\alpha^2 \simeq 100$.

If we use the same method as in section III, for reduction of the above determinant, we obtain the quadratic equation

$$(1 - \sigma^2)x^2 - 2G''x + H'' = 0$$

where
$$G'' = \frac{1 + \frac{1}{4} - \sigma - i\Theta(\frac{1}{2} - \sigma)}{1 - \psi_{10}} + \frac{k}{2} + \sigma - \frac{1}{4} + \frac{i\Theta}{2}$$

$$H'' = \frac{1 + 2k}{1 - \psi_{10}} - 1$$

For $\sigma = 1/2$, the effect of the term $i\Theta/2$ in the expression for G" (leaving aside the substitution of $\psi_{10}(\beta_0)$ for $F_{10}(\alpha)$, which is discussed below) will be to reduce the imaginary part of G", and therefore to reduce the damping of the wave in transmission, if the steady stream is in the same direction as the velocity of propagation. If the steady stream is in the opposite direction, damping will be increased. In the limiting condition of heavy loading and very stiff constraint, $(k \to -\infty)$, the viscous drag of the steady stream will have no effect, as might be expected.

The effect of substituting $\psi_{10}(\beta)$ for $F_{10}(\alpha)$ in the frequency equation may be studied as follows for the limiting condition of heavy loading and very stiff constraint. When $k \to -\infty$, corresponding to the earlier relation, $x = 2/(1 - F_{10})$, we write $x = 2/(1 - \psi_{10})$ so that

$$(1 - \sigma^2)\frac{x}{2} = \frac{(1 - \sigma^2)(1 - \frac{2\bar{w}_0}{c_1})}{1 - F_{10}(\beta_0) + (\frac{2\bar{w}_0}{c_1})S_{10}(\beta_0)} \qquad (10\text{-}44)$$

From equation 10-45 we may, as in section III, calculate the ratio of the wave velocity to that of the perfect fluid, c_1/c_0, and the attenuation factor, $\exp\left[-\frac{2\pi y}{x}\right]$.

The variation of c_1/c_0 for the particular value $\sigma = 1/2$, $k \to -\infty$ and $2\bar{w}_0/c_1 = 0.06$ (as in McDonald's experiment) is shown in figure 57, with the corresponding plot for $\sigma = 1/2$, $k \to -\infty$ with no steady component for comparison. We observe that the presence of the steady stream raises the wave velocity by 6% to 8%.

The variation of $\exp\left[-\frac{2\pi y}{x}\right]$ with respect to α is shown in figure 58. We observe that the damping of the wave in transmission is practically unchanged by the presence of the steady stream. It appears to be very slightly increased. This effect is opposite from that predicted by Morgan and Ferrante (1955) but in view of the widely different conditions, is not in conflict with their conclusions.

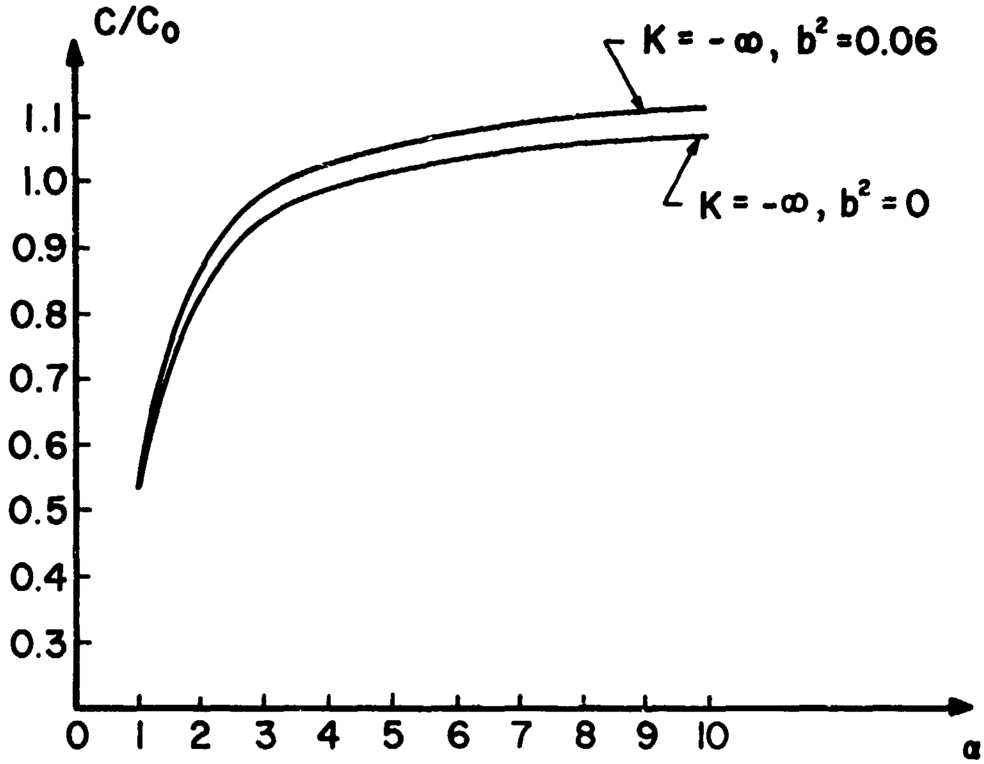

Figure 57. Comparison of variation in wave velocity with α for a steady stream of axial velocity 6% of the wave velocity with that for no steady stream.

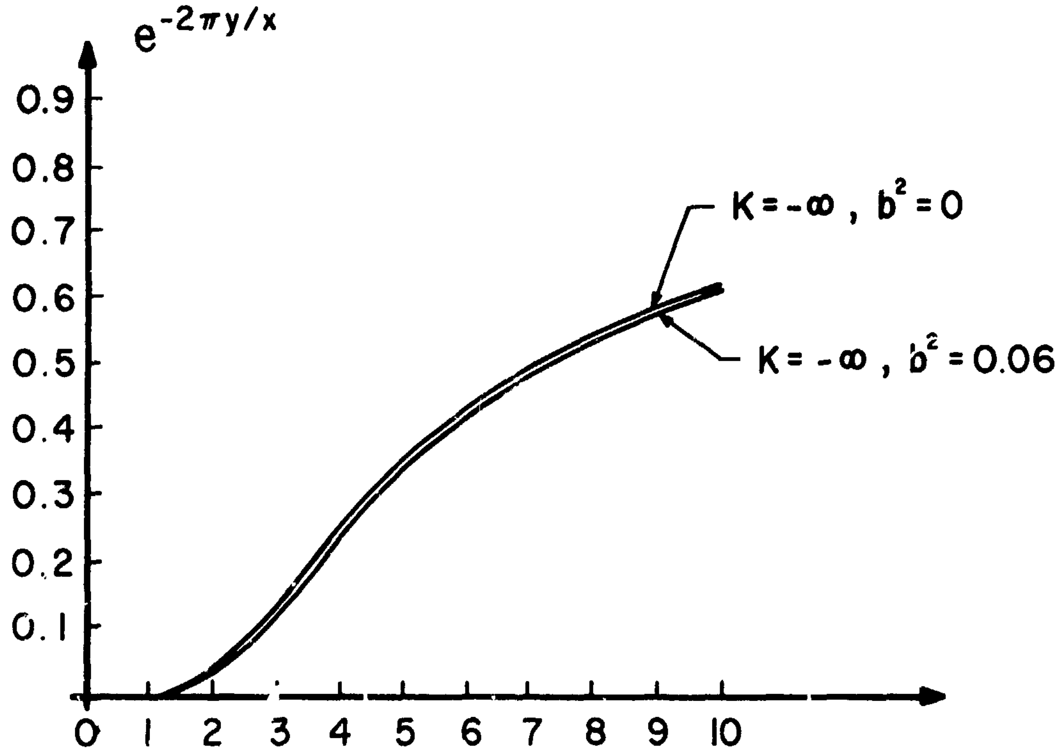

Figure 58. Comparison of the amount of damping in transmission, in presence and in absence of a steady stream.

For larger values of the factor $2\bar{w}_0/c$, this very simple approximation breaks down, and there would seem to be no alternative to a full-scale tabulation of the required solutions of the Confluent Hypergeometric equation and an attack on the problem in full generality. Before this can be contemplated, we need measurements of the comparative magnitude of the steady and oscillatory components of flow in the major arteries, together with accurate measurements of pulse velocity over short distances, in order to delimit the ranges of the parameters.

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